Pure Pursuit of a Target on a Circular Trajectory

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This investigation seeks to obtain estimates on the intercept time associated with a fast Pursuer implementing Pure Pursuit against a Target whose trajectory is a circular arc. The analytical solution of the kinematic equations of motion is not feasible and thus the baseline approach for computing the intercept time is numerical simulation, i.e., numerical integration of the kinematics. This method can obtain the intercept time in (essentially) real time for a single Pursuer against a single Target. Here, the goal is to estimate the intercept time using orders of magnitude less computational time. Although not explicitly explored in this work, the purpose of such approximations is to make feasible the rapid evaluation of Pursuer-to-Target assignments in a massive many-on-many engagement scenario wherein the intercept time may be a key measure. Several approximation methods are proposed here and their merits (or lack thereof) are demonstrated through simulation. Ultimately, assuming the Target is not actually turning may be suitable when its turn radius is relatively large, while assuming intercept occurs at the closest point on the turn circle may be suitable when its turn radius is relatively small.

I. Nomenclature

CC	=	Collision Course
LOS	=	line-of-sight
PN	=	Proportional Navigation
PP	=	Pure Pursuit
C	=	Target turn circle center
d	=	distance from Pursuer to Target
d_c	=	capture radius of Pursuer (assumed to be 1)
P	=	Pursuer position
T	=	Target position
κ	=	Target turn direction
μ	=	ratio of Target to Pursuer speed
ρ	=	Target turn circle radius
ψ	=	Target heading w.r.t. LOS

II. Introduction

Pursuit and evasion has long been a subject of study amongst mathematicians [1–4], theoretical biologists [5] (especially in the context of predator-prey interactions), and electrical engineers, especially in the context of missile guidance [6–8]. Additionally, pursuit and evasion serves as one of the primary examples and applications of differential game theory [9, 10]. In this paper, a scenario is considered in which a Target vehicle moves with a constant speed along a circular arc while a faster Pursuer vehicle, employing the Pure Pursuit (PP) geometric rule, approaches and intercepts the former with a finite capture radius. In PP, the Pursuer is always aimed at the instantaneous position of the Target (i.e., its heading is aligned with the line-of-sight (LOS) to the Target). Both of the vehicles' guidance strategies are fixed and

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thus the main aim of this study is to compute the time-to-go for the intercept. The analytical study of the equations of motion are not particularly fruitful, and thus the bulk of the discussion centers around possible approximation methods and their strengths and weaknesses under a variety of circumstances. Ultimately, the desire for fast approximations for the intercept time in this scenario stems from a desire to solve the so-called *Weapon Target Assignment* (WTA) problem in a many-Pursuer, many-Target engagement. In the WTA problem, an assignment of Pursuers to incoming Targets must be made according to some measure of cost (c.f. [11] for a general survey). For most applications, the time-to-go is a critical measure in the overall cost of an assignment [12] (in addition to control effort, risk, probability of kill, e.g.).

It is worth mentioning that the application considered herein is missile intercept. PP is among the most widely implemented guidance strategies for this application [13]. One may also consider the Proportional Navigation (PN) guidance strategy in this context as it is also widely implemented [7, 8]. In PN, the Pursuer commands its lateral acceleration to be proportional to the rate of change of the LOS angle. For PN, many analysis techniques have been used such as linearization, terminal projection, etc., which can aid in obtaining the intercept time. However, for PP, most works focus on cases for which the true intercept time may be obtained analytically (i.e., point capture of a Target on a constant course). The seminal books [7, 8] cover PN and many variants and extensions in detail whereas the book by Shneydor [6] covers PN in addition to PP.

The term *pursuit curve* was proposed by [1] to describe the path taken by an object moving with constant speed in PP of another object moving with constant speed. One of the earliest incarnations of the scenario involves a Pursuer who is initially *abeam* of the Target (i.e., at an angle $\pi/2$ w.r.t. the Target's heading) [1]. The same case was later analyzed in [1], although that work incorrectly concluded that the pursuit curve ends up being a parabola [14]. Much later, the PP intercept time for a general constant course angle for the Target was given in [15] by analytically solving the governing differential equation. Utilizing this result, the work [13] compared the expected intercept time for PP and collision course (CC) in the case that the Target's initial heading is drawn from a uniform distribution. CC refers to the Pursuer taking a heading such that the LOS does not rotate. When the Target is relatively slow, the two guidance strategies yield very similar intercept times. Finally, the explicit rectangular Cartesian equation for the Pursuer's trajectory to a Target on a constant course (i.e., the pursuit curve, itself) was given in the series of works [14, 16, 17].

Perhaps the next logical step after analyzing the case of a Target moving on a constant course is to consider the Target to move along the arc of a circle. The former assumes constant (inertial) heading while the latter assumes constant lateral acceleration for the Target (which is often a control input or reference command for missile systems). According to [6, 18], the problem of PP of a Target on a circular trajectory first appeared in *Ladies' Diary* in 1742 (shortly after the paper [1] appeared). Incidentally, this problem and the previous case of Target on a constant course were some of the main examples in a mathematics book entitled *Introduction to Nonlinear Differential and Integral Equations* [3]. A special case (referred to as the dog and duck problem) proposed by Hathaway [19] places the Pursuer at the origin of the Target's turn circle; the main aim was to determine the intercept time. Some later studies have been done, mainly employing numerical simulation (integration) to obtain results [20]. Shneydor's book [6] summarizes these works and provides some general observations for the point capture case. It is also worth mentioning that this scenario appears in Nahin's book [21] along with other scenarios such as cyclic pursuit in which a sequence of Pursuers all pursue one another.

Most of the literature referenced above pertains only to the case of point capture (i.e., the Pursuer must be at the exact same point in space as the Target to effect intercept). Some recent works have considered pursuit-evasion scenarios wherein the Pursuer has a finite (i.e., non-zero) capture radius. Reference [22] analyzed the case of PP of a Target on a constant course (along the lines of [14]) but with finite capture radius. One of the main results was a semi-analytic approach for computing the intercept time which, unlike in the point capture case, has no closed-form analytic solution. The case of an Evader who maximizes its intercept time against two Pursuers implementing PP was considered in both [23, 24]. Reference [23] proposes a suboptimal Evader strategy which equalizes the intercept time of both Pursuers using the expression from Shneydor [6] for point capture. That strategy was suboptimal due to two reasons: 1) the optimal Evader strategy is generally curved and 2) it did not account for the effect of the capture radius. Both of these aspects were analyzed in detail in [24] which numerically computes the true optimal Evader trajectory by utilizing the first-order necessary conditions for optimality. Finally, the paper [25] provides conditions for guaranteed capture, guaranteed escape, and evasive headings when the Pursuer has a finite capture radius and its range (i.e., maximum travel distance) is limited. Note that, unlike the other works in this group, the Pursuer in [25] is assumed to always take a CC trajectory instead of PP.

There are also several works which consider a Pursuer navigating to a circle, but, unlike all the aforementioned works, the Pursuer is turn-rate-constrained (i.e., a Dubins vehicle) and thus cannot physically achieve the PP guidance strategy. The min-time path for this Pursuer model to reach any point on a circle is covered in [26]. This result is extended to the case in which the Pursuer must reach a Target point which moves along the circle at a constant speed in [27, 28]. Another requirement in these works was for the Pursuer to intercept the Target with a heading that is tangent to the Target's turn circle.

The contributions of this paper are summarized as follows. Several approximation methods are proposed for estimating the intercept time for a Pursuer implementing PP to a Target moving on a circular trajectory including numerical integration, assuming constant closing speed (CC-like), assuming the Target is not turning, assuming the Target is not turning and the Pursuer uses CC, and assuming that the Target's turn radius is very small. The timely estimate of the intercept time for this case is an important aspect of the WTA problem (c.f. [11, 12]) for massive N-pursuer, M-Target types of scenarios wherein simulation of the equations of motion for the many possible assignments may be computationally prohibitive. This work also analyzes the quality of the approximations and identifies in what situations each may be appropriate or useful. Additionally, the cases in which the approximations perform poorly are demonstrated. Lastly, some computational benchmarks are given so that the tradeoff between accuracy and compute time may be observed.

The remainder of the paper is organized as follows. Section III introduces the kinematics and discusses the difficulties of directly analyzing the differential equations. Section IV proposes several methods to approximate the intercept time which vary in accuracy and computational effort required. Section V shows several simulations illustrating some general trends as well as a more detailed investigation of the error of the constant course approximation. Section VI summarizes the compute times required for the methods. Finally, the paper is concluded in Section VII with some general recommendations for computing intercept times for this scenario.

III. Kinematics

Let ρ be the radius of the Target's turn circle and $\kappa \in \{-1, 1\}$ be the direction of the turn. When the Target is turning counter-clockwise, for example, $\kappa = 1$. The kinematics in the relative frame are

$$\begin{bmatrix} \dot{d} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \mu \cos \psi - 1 \\ -\frac{\mu}{d} \sin \psi - \kappa \frac{\mu}{\rho} \end{bmatrix}, \tag{1}$$

where μ is the Target's linear speed (assumed to be < 1, i.e., less than the Pursuer's speed of 1) and ψ is the angle of the Target's motion w.r.t. the line of sight (LOS) from the Pursuer (clockwise is positive). See Fig. 1 for an illustration of the scenario. Intercept is said to occur when $d = d_c$, the Pursuer's capture radius. One may, without loss of generality, set $d_c = 1$ (i.e., by normalizing distances by the Pursuer's capture radius). It is assumed henceforth that $d \geq 1$, that is the Target is never inside the Pursuer's capture radius. Reference [6] quotes [3] as saying that this governing equation "presents unusual difficulties".

When $\rho \to \infty$ we recover the kinematics from [22] (as expected):

$$\begin{bmatrix} \dot{d} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \mu \cos \psi - 1 \\ -\frac{\mu}{d} \sin \psi \end{bmatrix}. \tag{2}$$

Regarding this particular case, the system of differential equations can be solved in order to obtain an equation which relates d_0, ψ_0 with d_f, ψ_f. By setting d_f to the capture distance, one can numerically solve for ψ_f quickly and accurately thanks to the monotonicity of ψ. The ψ_f can then be used to obtain the intercept time via an equation from [6] (also courtesy of the fact that ψ monotonically approaches 0 as time goes on).

Now, regarding the case where the Target is moving on a circle, (1), we do not enjoy the same property of ψ tending towards 0 as d approaches the capture distance. As for the capture distance, without loss of generality, we consider it to be the unit distance. Looking at (1), it is tempting to say that ψ *is* monotonic as long as $d > \rho$ (in which case the $\frac{\mu}{\rho}$ term dominates the sign of $\dot{\psi}$). However, this is misleading since it only means that $\dot{\psi}$ has a particular sign and thus ψ, itself may, for example, run all the way through π to $-\pi$ and then wrap back to π. Thus, the issue, here, is that ψ does not necessarily tend towards 0 in general. It is for this same reason that even if one could solve the system (1) to relate d_0, ψ_0 with d_f, ψ_f there is no obvious way to obtain the associated final time. Indeed, the system (1) is *not* easily solvable due to the fact that $\frac{\partial d}{\partial \psi}$ cannot be factored into a function of ψ multiplied by a function of d (as was the case for (2)). Of course, one can alter (1) in a myriad of ways to recover this useful property. For example, the term $\frac{\mu}{\rho}$ in $\dot{\psi}$ may be divided by d (or some constant times d). In that case, an analytic solution to the system of differential equations exists. However, the numerical solution of ψ_f is considerably more difficult than in the case of (2) due to the possibility that ψ is not monotonic. Then there remains the issue of having no obvious way to obtain t_f from ψ_f.

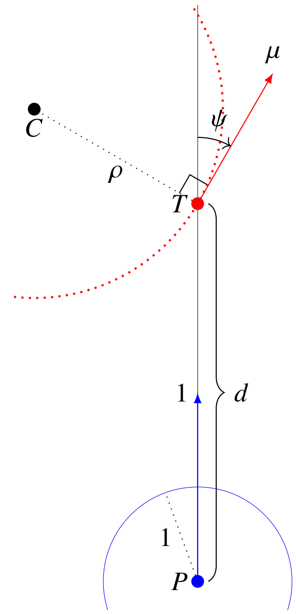


Fig. 1 Schematic illustration of the scenario

IV. Approximations

In the following subsections, several different intercept time approximation methods are proposed. It should be noted that although intercept time is the main concern one may also back out the estimated location of the Target at intercept given its initial position, speed, turning direction, and radius. There are two main criteria for the quality and/or general usefulness of these approximations: 1) required computational time and 2) the error of the approximation w.r.t. the true intercept time. Both of these aspects are investigated in the following sections.

A. Numerical Integration

The system (1) may be numerically integrated until the condition $d = d_c$ (i.e., intercept) has occurred. This condition is guaranteed to occur by way of d being monotonic (thanks to the assumption that P is faster, i.e., $\mu < 1$). The computational effort required can be tuned, for example, by using a fixed-step integration scheme and altering Δt. There is, however, no obvious way to guarantee a particular number of integration steps without knowing t_f (which is what we are trying to find in the first place). Nonetheless, this approach provides a reliable and accurate approximation of the true t_f with many parameters in which accuracy and computational effort may be traded off. Any other approximation method proposed herein must require significantly less computational effort for it to supersede numerical integration in usefulness.

B. Assume Constant Closing Speed

One extremely naive approximation is to measure the closing speed between P and T (i.e., $\dot{d}$) in the current configuration and compute t_f by assuming this speed remains constant throughout the engagement. In general, for this to actually be true, the Target's trajectory must be some kind of spiral, i.e., such that ψ remains essentially constant. The final time, assuming a constant closing speed, is denoted with subscript v:

$$t_v\left(d, \psi\right) = \frac{d(t)}{-\dot{d}(t)} = \frac{d}{1 - \mu \cos \psi} \tag{3}$$

C. Assume Target is not Turning

Another naive sort of approximation is to ignore the Target's turning altogether and employ the intercept time associated with the Target moving on a constant course from [22] (see also [6, Eq. 3.9]). In terms of computational effort, this more expensive than the previous approximation when the capture radius of the Pursuer is non-zero since a root-finding method is required. This assumption is obviously most appropriate when ρ is sufficiently large; it becomes exact when $\rho \to \infty$, as mentioned earlier. The approximated final time under this assumption is denoted with subscript $\rho\infty$ for "$\rho = \infty$":

$$t_{\rho\infty}(d,\psi) = \frac{d}{\mu}\frac{\left(\frac{1}{\mu} + \cos\psi - \frac{d_c}{d}\left(\frac{1}{\mu} + \cos\psi_f\right)\right)}{\frac{1}{\mu^2} - 1}, \tag{4}$$

where ψ_f is a quantity obtained by solving the system (2) via root-finding.

D. Assume Target is not Turning and Pursuer Takes Collision Course

This assumption is similar to the previous but with the added assumption that the Pursuer takes the CC trajectory with the (assumed) non-maneuvering Target. Adding the CC assumption results in two benefits: 1) its intercept time is a lower bound for $t_{\rho\infty}$, and 2) its intercept time is given by an analytical expression and is thus easier to compute than $t_{\rho\infty}$. Altogether, the final time based on this assumption is denoted with subscript cc for "collision course". The intercept time is obtained by creating a triangle wherein P_f is collinear with T_f and $\overline{TT_f} = \mu\overline{PP_f}$. Then, from the Law of Cosines, the associated intercept time is:

$$t_{cc} = \frac{d\mu\cos\psi - d_c + \sqrt{(d_c - d\mu\cos\psi)^2 - \left(1-\mu^2\right)\left(d_c^2 - d^2\right)}}{1-\mu^2}. \tag{5}$$

Note that the term $\left(1-\mu^2\right)(d_c - d)$ is always negative and thus the radical produces a real number. Moreover, this real number is greater than the term preceding it and thus the positive version of the quadratic equation is taken in order to make $t_{cc} = \overline{PP_f}$ positive.

Scharf *et al.* provide the CC (i.e., intercept course) intercept time for point capture. Incidentally, for CC with point capture, the LOS does not rotate which keeps the range rate constant, and thus the intercept time computation resembles (3).

E. Assume Target Turn Radius is Negligible

Obviously, the two preceding assumptions ought to be most suitable when the Target turn circle radius, ρ, is large. This assumption, on the other hand, is designed to provide a good approximation in the case that ρ is small. In the limit where $\rho \to 0$, the Target is effectively stationary. Here, rather than go all the way to this limit, ρ, however small it may be, is accounted for. It is assumed that the Pursuer takes a straight line path such that its terminal position is collinear with the Target turn circle center, C, and the Target happens to be on the closest point of its turn circle to the Pursuer. The last piece, along with the straight-line assumption, ensure that this approximation always underestimates the true intercept time. This assumption is denoted with the subscript $\rho 0$ (i.e., "$0 < \rho \ll 1$").

$$t_{\rho 0} = \overline{PC} - \rho - d_c \tag{6}$$

Of course, when the Pursuer's position lies inside the Target's turn circle, $t_{\rho 0}$ is negative. Thus, in practice, one should take the max $\left(t_{\rho 0}, 0\right)$.

V. Simulations

In this section, the aforementioned intercept time approximations are simulated under a variety of initial conditions and parameter settings. For the purposes of visualization, the system (1) can be transformed into Cartesian (x, y) coordinates as follows. As this transformation is non-unique, the initial Target position is chosen to be the origin and the Pursuer's initial position is chosen to be $(0, -d_0)$. The Target's turn circle center (c.f. Fig. 1) is

$$C = \rho\begin{bmatrix}\cos(-\psi_0) & \sin(-\psi_0)\end{bmatrix}^\top. \tag{7}$$

Define

$$\phi(t) = \frac{\pi}{2} - \psi_0 + \frac{\mu}{\rho}t. \tag{8}$$

Then the Target and Pursuer trajectories in Cartesian coordinates are

$$
\begin{aligned}
T(t) &= C + \rho \left[\cos\left(-\tfrac{\pi}{2} + \phi(t)\right) \quad \sin\left(-\tfrac{\pi}{2} + \phi(t)\right)\right]^\top \\
P(t) &= T(t) - d(t)\left[\cos\left(\phi(t) + \psi(t)\right) \quad \sin\left(\phi(t) + \psi(t)\right)\right]^\top .
\end{aligned} \tag{9}
$$

A. Qualitative Discussion

For all of the following examples the Target begins at the origin and moves with a speed of $\mu = 0.8$, the Pursuer begins at $(x, y) = (0, -5)$ and moves with a speed of 1, and the capture radius $d_c = 1$. The figures in this section display the example trajectory in Cartesian coordinates on the left and a comparison of the times-to-go throughout the duration on the right. In the time-to-go plots, the initial time-to-go value is shown as a dashed line with slope of -1 while the solid lines indicate the re-computed time-to-go value over the course of the simulation.

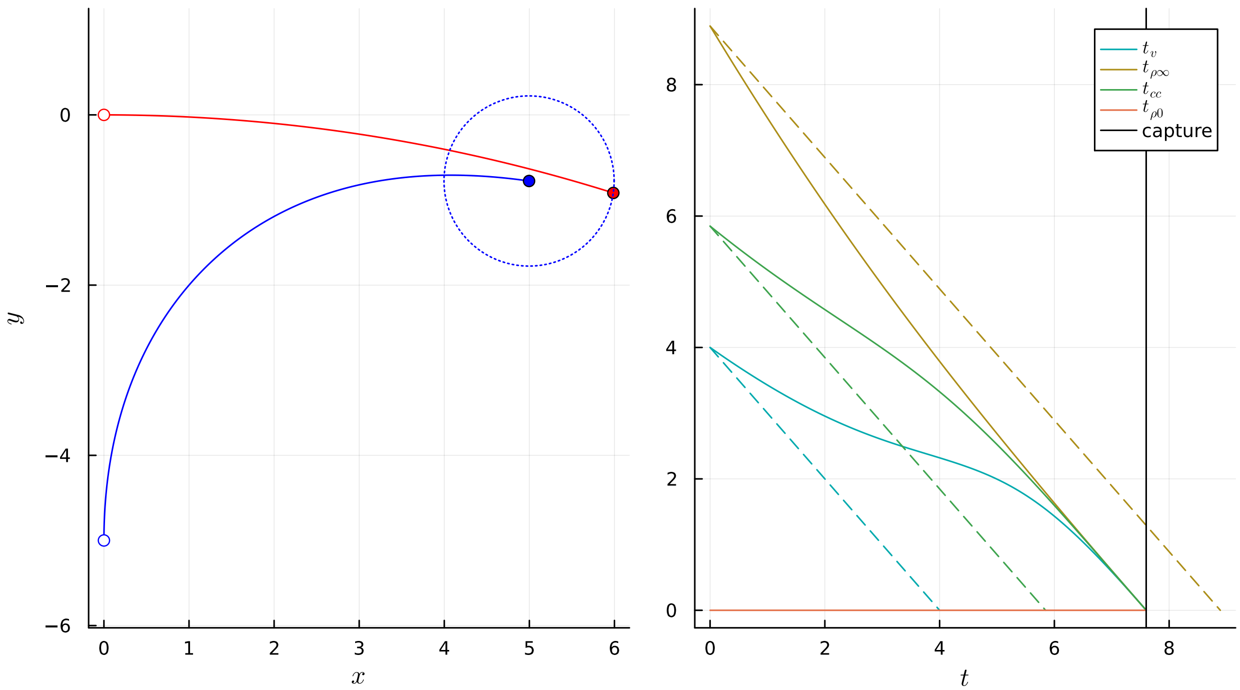


Fig. 2 Large ρ, Target turning "towards" Pursuer.

As can be seen in Fig. 2 and Fig. 3, when ρ is large in comparison to d_0 and d_c the constant course approximation, $t_{\rho\infty}$, gives a decent estimate of the true intercept time. Moreover, because the arc angle traversed by the Target is small in these examples, there is an additional property that in the case of Target turning "towards" the Pursuer $t_{\rho\infty}$ *overestimates* the true intercept time whereas when the Target turns "away" $t_{\rho\infty}$ *underestimates* the true intercept time. In general, this property may not hold, particularly when the arc angle traversed by the Target is large. This is because, even in the case that the Target turns away, initially, eventually it will be turning towards the Pursuer. Based on Fig. 2 the collision course approximation, t_{cc}, appears to be a reasonable approximation as it is about as accurate as $t_{\rho\infty}$. However, in Fig. 3, when the Target is turning away, $t_{\rho\infty}$ serves as a lower bound to t_f, and thus its own lower bound, t_{cc}, is even further away from the true intercept time.

From Fig. 4, when ρ is large and ψ is small, the constant closing speed approximation, t_v, turns out to give the best approximation. In this configuration, the Pursuer is almost always directly behind the Target throughout the trajectory and thus the Target's turning keeps ψ close to constant. Again, $t_{\rho\infty}$ is not a bad approximation, here, as in the other large ρ cases.

When ρ is sufficiently small in comparison to d_0 and d_c, the Pursuer's trajectory essentially exhibits small oscillations around heading toward the Target's turn circle center. Also, in this situation, there is little variation in the length of the

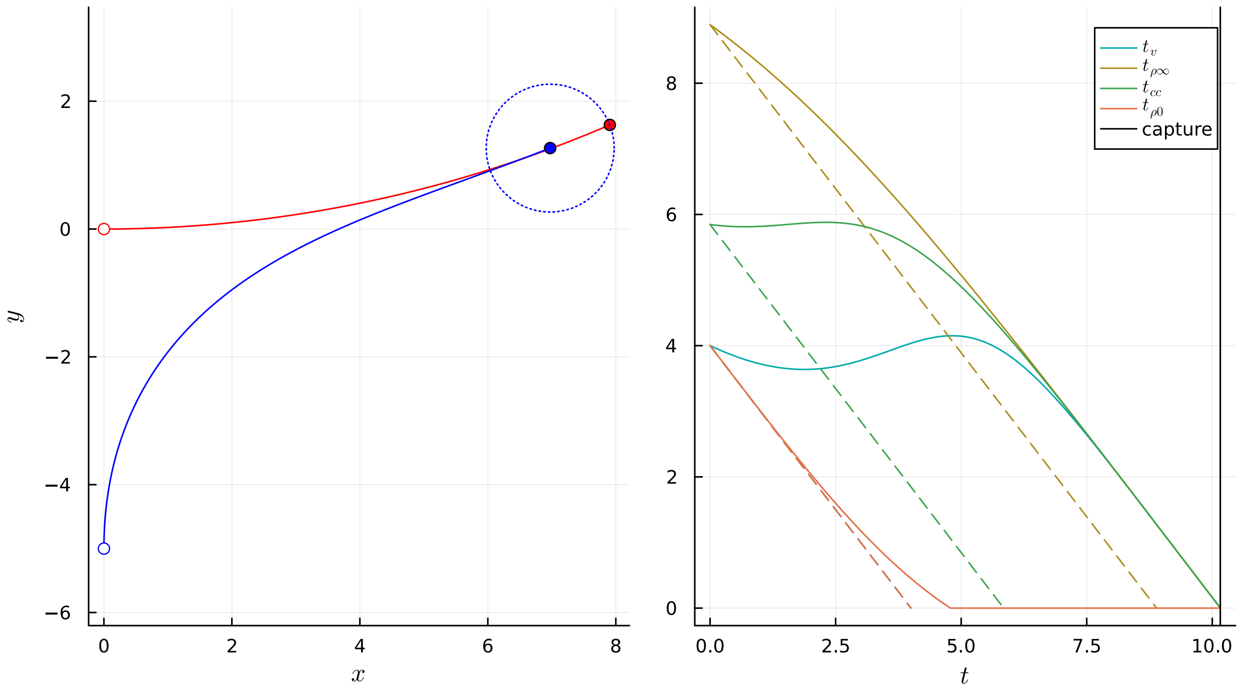


Fig. 3 Large ρ, Target turning “away” from Pursuer.

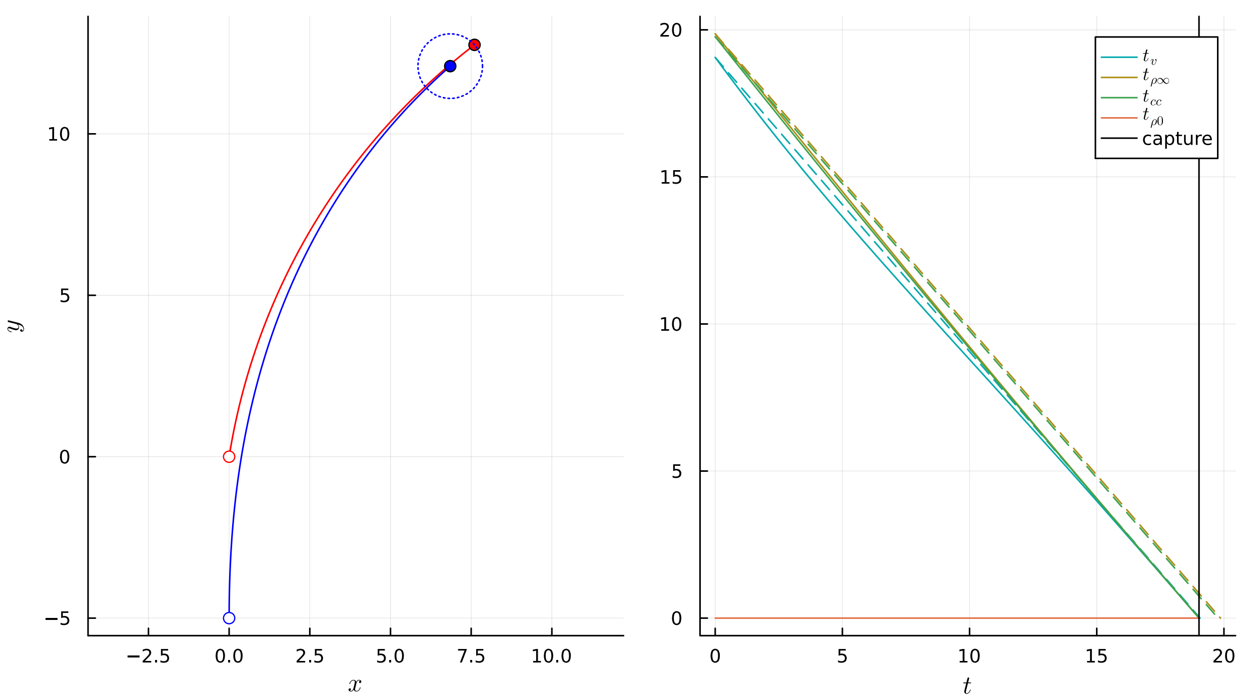


Fig. 4 Large ρ with small initial ψ.

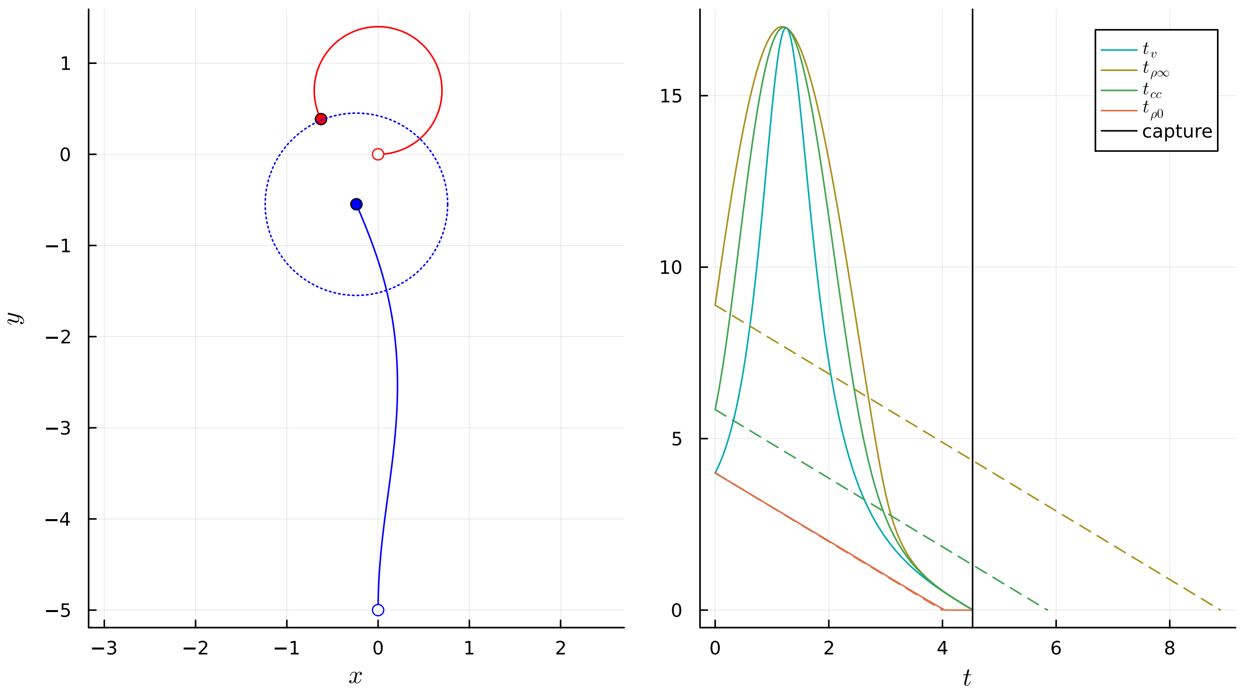


Fig. 5 Small ρ.

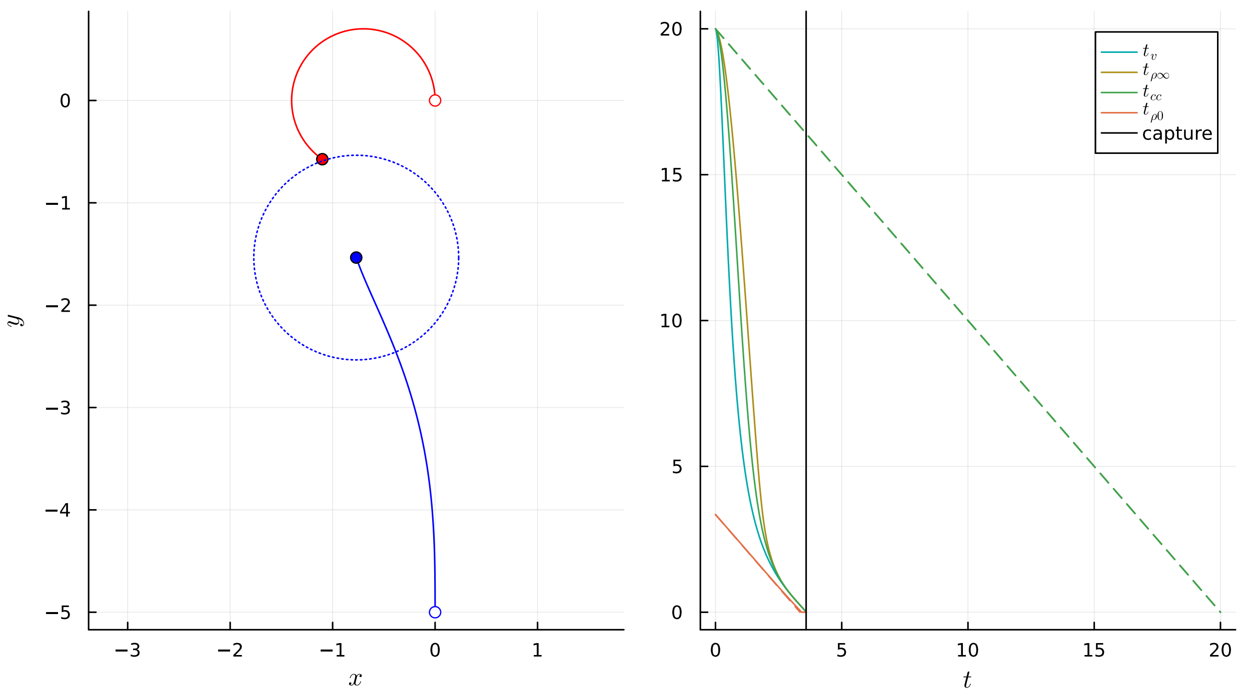


Fig. 6 Small ρ with small ψ.

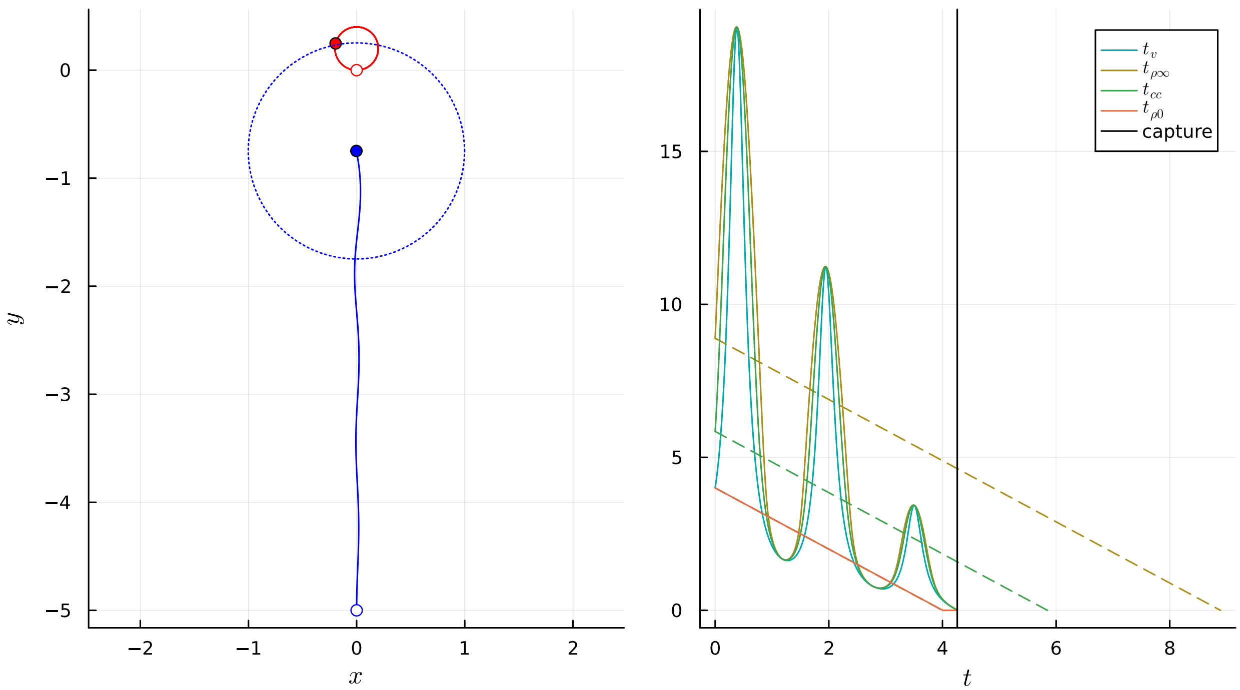


Fig. 7 Very small ρ.

straight-line path joining the Pursuer to any point on the Target's turn circle. Therefore, since the Pursuer's trajectory is *approximately* straight, the small ρ approximation, $t_{\rho 0}$, is very good. The approximations t_v and t_{cc}, on the other hand, are generally very bad in the case of small ρ. This is due to ψ varying over a large range throughout the trajectory. As a result, these approximations oscillate with a very large amplitude. These trends are apparent in Figs. 5 to 7. In particular, Fig. 6 highlights how bad the constant closing speed, non-turning, and collision course approximations can be since, when $\psi = 0$, they all yield the Pure Evasion intercept time (i.e., the worst case intercept time if the Target maneuvers adversarially) which is a grossly conservative overestimation in this example.

B. Error of the Constant Course Approximation

In this section, the error of the constant course approximation w.r.t. the true intercept time is discussed. The intercept time obtained via numerical integration is treated as the truth data. Error is defined as:

$$\text{error} = \frac{t_{\rho\infty} - t_f}{t_f}, \tag{10}$$

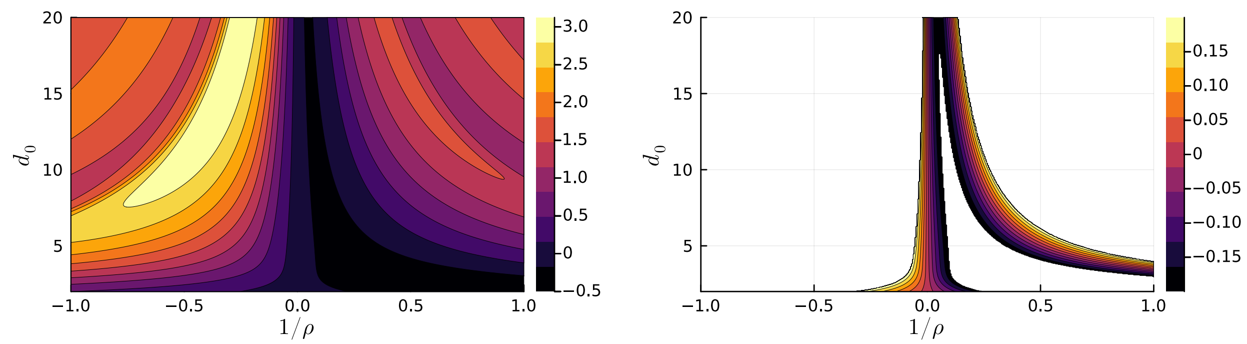
and thus positive error is associated with an over approximation of the intercept time while negative error is associated with an under approximation. Without loss of generality, the capture distance is treated as the unit distance. The intercept time (and thus error) depends on the target turn circle radius and direction, ρ and κ, respectively, the speed ratio, μ, and the initial state (d_0, ψ_0). Since the constant course approximation makes the assumption that $\rho = \infty$ the error is very sensitive to the ρ parameter; smaller error is expected for larger ρ and *vice versa*. In order to examine the effects of each of the variables, a nominal configuration is simulated over a two-dimensional slice. Throughout the paper, ρ and κ are combined into the former, i.e., positive ρ indicates a CCW Target turn while negative ρ indicates CW. The nominal configuration is listed in Table 1.

Figs. 8 to 10 show how the approximation error varies for slices in the d_0, ρ, ψ_0, ρ, and μ, ρ directions, respectively. In these figures, the abscissa is $1/\rho$ and so values near 0 correspond to very large Target turn circle radii wherein the constant course approximation should be very good.

A few trends are apparent. As expected, when $1/\rho$ is very small, the error is small (c.f. Fig. 8b). When ψ_0 is close to π (i.e., when the Target is nearly pointing at the Pursuer), the approximation error is not very sensitive to the Target turn radius (c.f., Fig. 9b). When μ is small (i.e., the Pursuer is much faster than the Target), the approximation error is

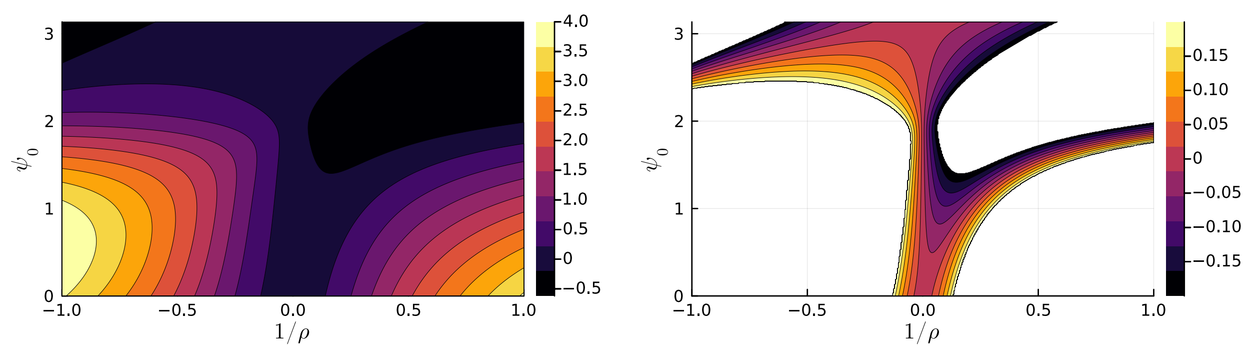
Table 1 Nominal configuration for constant course error sweep.

Variable	Nominal Value
d_0	5
ψ_0	$\pi/2$
μ	0.8
ρ	-20



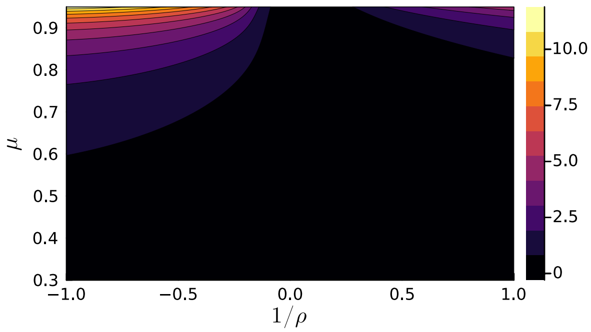
(a) **(b)**

Fig. 8 Relative error of the constant course approximation for a slice in the d and ρ variables (a) and a masked version (b) which shows more detail for the region in which the error is less than 20%.

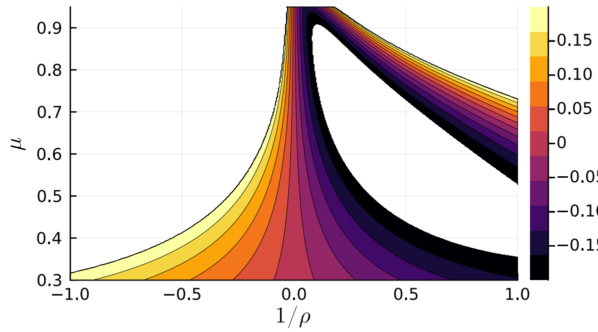


(a) **(b)**

Fig. 9 Relative error of the constant course approximation for a slice in the ψ and ρ variables (a) and a masked version (b) which shows more detail for the region in which the error is less than 20%.



(a)



(b)

Fig. 10 Relative error of the constant course approximation for a slice in the μ and ρ variables (a) and a masked version (b) which shows more detail for the region in which the error is less than 20%.

not very sensitive to the Target turn radius (c.f., Fig. 10b). The worst error occurs when μ is close to 1 and the Target is turning CW (i.e., turning away in the nominal configuration, c.f., Fig. 10a).

VI. Computational Benchmarks

This section provides some computational benchmarks for some of the different methodologies. In general, the methodologies can be categorized into 1) numerical integration, 2) semi-analytic (which includes the constant course approximation), and 3) analytic (which includes the constant closing speed, collision course, and small radius approximations). Within the analytic category, the collision course approximation requires the most operations (generally) and thus will serve as a conservative representative of this category.

The methods were implemented in the Julia programming language (version 1.8) and run on a MacBook Pro with a 2.2 GHz 6-Core Intel Core i7 processor, and 16 GB 2400 MHz DDR4 RAM. The numerical integration implementation utilized the `DifferentialEquations.jl` package with an absolute and relative tolerance set to 1e-12. For the semi-analytic implementation (constant course approximation), the absolute tolerance (on ψ_f) was set to 1e-6.

Table 2 shows a comparison of the compute times for the example in Fig. 3 at the initial condition (i.e., $t = 0$). As expected, the compute times are highest for numerical integration and lowest for the analytic method(s) (by several orders of magnitude, respectively). The application will dictate whether one or more are appropriate to utilize depending on the desired accuracy and computational and/or time budget.

VII. Conclusions

This report investigated the computation of the intercept time associated with a Pursuer implementing Pure Pursuit chasing a Target moving on the arc of a circle with known radius. Obviously, for such a simple kinematic model, numerical integration may be used to obtain a solution without much difficulty. Several approximation methods were proposed and tested qualitatively as alternative means of computing/estimating the intercept time. They were: the constant closing speed assumption, constant course assumption, collision course assumption, and a small radius assumption. The solutions based on the constant course assumption require numerical solution of a function (e.g., via root-finding) whereas all of the others are analytic and thus far less computationally expensive. Except for some

Table 2 Compute times for intercept time estimates for Fig. 3 at $t = 0$.

Methodology	Mean Time (s)	Std. Dev (s)	Samples
Numerical Integration	6.50e-4	3.35e-3	7,663
Semi-Analytic	9.74e-6	1.40e-6	10,000
Analytic	2.53e-7	7.00e-7	10,000

very particular cases (such as in the Pursuer starting behind the Target w.r.t. the latter's heading), the constant closing speed assumption does not yield the best approximation to the true intercept time. When ρ is relatively large, the constant course assumption can actually be quite good and it has the added benefit of providing a pseudo upper or lower bound depending on the Target's direction of turning (when the arc swept by the Target is sufficiently small). When ρ is relatively small, the small radius assumption is not only the best, but generally a fairly close approximation of the true intercept time. Additionally, this approximation has the added benefit of providing a lower bound on the true intercept time (always). There appears to be a general trend, as expected, that the approximations which require the least amount of computation tend to have the worst error. The constant course assumption is the recommended approximation when the Target turn circle radius is relatively large, whereas the small radius assumption should be used otherwise. The former has reasonable error especially when the Target is much slower than the Pursuer and/or the Target is close to head-on. Future research directions include utilizing these intercept time approximations in large multi-Pursuer, multi-Target simulations in order to assign Targets to Pursuers.

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