Range-Limited Pursuit-Evasion

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***Abstract*—In pursuit-evasion the objective of the pursuer is to capture the evader. In this work, the faster pursuer is modeled to have limited range and therefore optimal strategies for the pursuer and evader change. Depending upon the range limits of the pursuer the evader may evade capture by the pursuer. This paper describes the optimal strategies and nuances that appear for point-capture or when the pursuer is endowed with a non-zero capture radius.**

***Index Terms*—pursuit-evasion, differential games, multi-agent systems**

I. Introduction

Pursuit-Evasion considers two parties labeled as either "pursuers" or "evaders." In the general sense, the pursuers strive to capture the evaders while the evaders strive to escape capture by the ensuing pursuers. The conflict that exists between these two parties has been very well studied as a differential game [1]–[3]. While prior art has considered the pursuer to have unlimited resources, this paper considers a range-limited pursuer in an effort to provide necessary realism to the general class of 1-v-1 pursuit-evasion. In this case, the limit on the pursuer's range may be due to limited fuel onboard or even a limit on the distance it can travel from a fixed location (e.g., communication station or base). Two capture conditions for the pursuer are considered: capture through collocation (i.e., point capture) and capture when the pursuer is endowed with a non-zero capture radius.

Concerning point capture, Rufus Isaacs rightly pointed out that for a two-dimensional, planar engagement that point capture results in a degenerate zero-dimensional terminal surface whereas the terminal surface ought to be of one dimension less than the original scenario [1]. This is most obvious within the Homicidal Chauffeur Differential Game [4] wherein a turn-constrained agent pursues a slower agent who can turn on a dime. Point capture, in this case is not even a mathematical possibility as the evader can always side-step the pursuer (i.e., into the latter's turn-circle) and avoid capture. Nonetheless, some differential game solutions have been obtained in cases where all agents move with simple motion and the game ends in point capture [5].

A non-zero capture radius for an ensuing pursuer has been previously investigated in the context of pursuit-evasion scenarios with simple motion, one evader, and either one or two pursuers. Oyler described the dominance regions to be larger as a result of the pursuer having non-zero capture radius [6]. Fuchs *et al.* treated the two-pursuer single-evader (two cutters and fugitive ship game) wherein the fast pursuers have the same speed and a finite capture radius and obtained the solution, but not in closed-form [7]. Wasz also considered non-zero capture radius for this scenario [8], [9]. Casini *et al.* considered a special case of this scenario in which all three agents have the same speed [10]. Vlassakis *et al.* considered the case where the pursuers have different speeds [11]. Von Moll *et al.* investigated scenarios in which one or more pursuers implement the Pure Pursuit geometric rule and are have a non-zero capture radius [12], [13].

Several other pursuit-evasion scenarios have been considered in the context of non-zero capture radius. In a work by Fuchs *et al.* the active target defense scenario was solved for an attacker with a finite capture radius [14]. In the scenario, a pursuing attacker strives to capture a target vehicle that is teamed with a friendly defender. In [15], the Cartesian ovals for cases where the pursuer is endowed with a non-zero capture radius and the pursuer/evader speed ratios are greater than and less than 1 are presented. In [16] the containment of a faster evader is considered when multiple pursuers have identical non-zero capture radius. In all of these works, all parties are assumed to have unlimited range.

There is also a body of work concerning pursuit-evasion scenarios with some fixed limit - either a fixed final time, or constraints imposed on the pursuer. In the case that an agent takes a straight line path (which often occurs for extremal controls for simple motion agents) a fixed final time is equivalent to a limit on range. Along these lines, Chen *et al.* recently analyzed a reach-avoid pursuit-evasion game with a fixed time [17]. Some work has also been done on cases where the pursuers have either integral constraints on the control effort (which, again, could be mapped to fuel usage) and/or instantaneous constraints on control effort [18]. In addition, Bakolas *et al.* have considered the so-called relay pursuit scenario in which a team of pursuers are scattered over an area but only one pursuer actively pursues the evader at a given time. Although a range limitation is not explicitly mentioned there it's clear that such a group pursuit strategy allows the inactive pursuers to conserve their control effort (e.g. fuel). In addition to having limitations in range, time, or control effort, it could be the case that the pursuer has limited sensing. Reference [19], for example, considers a pursuer with self-triggered, intermittent sensing. For certain applications in which sensing is expensive one must consider the interplay between the sensing control (i.e., when to sense) and motion control (i.e., where to aim). Finally, the concept of a pursuers' reachable region can be useful as a mechanism for evader path planning (c.f., e.g., [20]).

This paper considers a range-limited, faster pursuer versus an evader, both moving with simple motion. It is assumed that the heading of the evader is known to the pursuer. The contributions of the paper are: 1) the optimal (i.e., minimum time) heading for the pursuer is shown to be a straight line, 2) the optimal pursuer heading and distance (which is analogous to time, in this case) are provided in closed form for the case of point capture, and 3) likewise for the case of non-zero capture radius. For either capture condition, there are three possibilities which arise as a result of the pursuer's range limit: 1) the pursuer's and evader's reachable regions do not intersect, meaning capture is not possible, or 2) the evader's reachable region is a subset of the pursuer's, meaning capture is inevitable, or 3) the intersection of their reachable regions is a subset of the evader's, meaning capture may or may not be possible depending on the evader's heading. In the last case, the analytic expressions for the critical evader headings for which capture is just barely possible are provided.

The remainder of this paper is organized as follows. Section II defines the vehicle dynamics and objectives surrounding the pursuit-evasion scenario where agents exhibit simple motion. In Section III, the locus of pursuer-evader interceptions, when the interception is defined through point-capture, is presented. Next, in Section IV the pursuer is assumed to have limited range and capture occurs through point-capture. After that, in Section V a faster pursuer endowed with a non-zero capture radius is considered. Next, in Section VI, the pursuer and evader strategies when the pursuer is range-limited and the pursuer is endowed with non-zero capture radius is presented. Examples are presented along with the derivations. Finally, in Section VII concluding remarks are made, highlighting the various aspects of these planar engagements.

II. Problem

Consider a pursuer, P and evader E. The pursuer aims to capture the evader in minimum time while the evader strives to prevent capture. The pursuer and evader in Cartesian coordinates are $(x_P, y_P) \in \mathbb{R}^2$ and $(x_E, y_E) \in \mathbb{R}^2$, respectively. The pursuer is faster than the evader and their speeds are v_P and v_E, respectively. In general, it is assumed that the pursuer is faster than the evader and therefore: $\mu = v_E/v_P$ is bounded, $0 < \mu < 1$.

The pursuer and evader are assumed to exhibit simple motion and the dynamics for the pursuit-evasion scenario is as follows:

$$\begin{aligned} \dot{x}_P &= v_P \cos\psi_P, \\ \dot{y}_P &= v_P \sin\psi_P, \\ \dot{x}_E &= v_E \cos\psi_E, \\ \dot{y}_E &= v_E \sin\psi_E. \end{aligned} \tag{1}$$

where the control for the pursuer is its heading, $\psi_P \in [0, 2\pi)$, and the control for the evader it its heading, $\psi_E \in [0, 2\pi)$.

A differential game develops between the pursuer and evader as described by Isaacs [1]. Because the pursuer and evader exhibit simple motion, it has been shown that their optimal strategies are straight-line trajectories.

The Hamiltonian for the resulting differential game is

$$\begin{aligned} \mathscr{H} = p_{x_P} v_P \cos\psi_P + p_{y_P} v_P \sin\psi_P + \\ p_{x_E} v_E \cos\psi_E + p_{y_E} v_E \sin\psi_E \end{aligned} \tag{2}$$

where $\mathbf{p} = (p_{x_P}, p_{y_P}, p_{x_E}, p_{y_E})^\intercal$ represent the costates.

The Pontraygin Minimum Principle (PMP) yields necessary conditions for optimality in the pursuit-evasion differential game.

$$\dot{\mathbf{x}}^*(t) = \frac{\partial \mathscr{H}(\mathbf{x}^*(t), \mathbf{p}(t), \psi_P^*(t), \psi_E^*(t), t)}{\partial \mathbf{p}}, \tag{3}$$

$$\dot{\mathbf{p}}(t) = -\frac{\partial \mathscr{H}(\mathbf{x}^*(t), \mathbf{p}(t), \psi_P^*(t), \psi_E^*(t), t)}{\partial \mathbf{x}}, \tag{4}$$

$$\mathbf{0} = \frac{\partial \mathscr{H}(\mathbf{x}^*(t), \mathbf{p}(t), \psi_P^*(t), \psi_E^*(t), t)}{\partial \psi_P}, \tag{5}$$

$$\mathbf{0} = \frac{\partial \mathscr{H}(\mathbf{x}^*(t), \mathbf{p}(t), \psi_P^*(t), \psi_E^*(t), t)}{\partial \psi_E}. \tag{6}$$

and $\mathscr{H}(t_f) = 0$. The superscript, $*$ represents optimality. Evaluating the necessary conditions in (4), the costates are found to be constant, as expected:

$$\dot{p}_{x_P}(t) = 0,\ \dot{p}_{y_P}(t) = 0,\ \dot{p}_{x_E}(t) = 0,\ \dot{p}_{y_E}(t) = 0. \tag{7}$$

Evaluating the partials in (5) and (6),

$$0 = -p_{x_P} \sin\psi_P^* + p_{y_P} \cos\psi_P^*, \tag{8}$$

$$0 = -p_{x_E} \sin\psi_E^* + p_{y_E} \cos\psi_E^* \tag{9}$$

In (7), the costates under play for the pursuer and evader are found to be constant. By (8) and (9) the optimal headings for the pursuer and the evader are found to be constant. Therefore, optimal strategies for the pursuer and evader are constant bearing trajectories – optimal strategies are straight-lines.

III. Point Capture

Consider a fast pursuer and slower evader. The pursuer captures the evader using point capture – the pursuer and evader are collocated at final time. When the state belongs to the point capture set, $\mathscr{C}_{pc} \triangleq \{\mathbf{x} | (x_P - x_E)^2 + (y_P - y_E)^2 = 0\}$, the evader has captured the evader. As shown earlier, since both the pursuer and evader exhibit simple motion (both agents may turn instantaneously), the costates under optimal play are constant by (7) and the resulting optimal pursuer and evader trajectories are straight lines. Recall the speed ratio: $\mu = v_E/v_P < 1$. The distance traversed by the agents (prior to point-capture) is proportional:

$$\overline{EI} = \mu \overline{PI}. \tag{10}$$

where E is the initial location of the evader, P is the initial location of the pursuer, and I is the location where the evader captures the pursuer, this is depicted in Fig. 1. Squaring both sides:

$$\overline{EI}^2 = \mu^2 \overline{PI}^2. \tag{11}$$

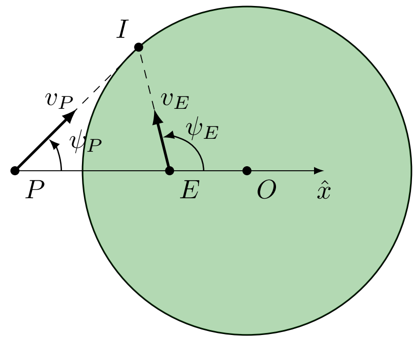

Fig. 1. The locus of point-capture interceptions of the evader by the faster pursuer (whose range is unlimited) is described by Apollonius circle.

Define the Cartesian coordinates whose x-axis is aligned from E to P, and E is at the origin $(0,0)$. Therefore, in this frame,

$$\begin{aligned}\overline{EI}^2 &= x^2 + y^2 \\ \overline{PI}^2 &= (d-x)^2 + y^2\end{aligned} \tag{12}$$

Substitution of (12) into (11), the following is obtained:

$$x^2 + y^2 = \mu^2((d-x)^2 + y^2). \tag{13}$$

Through algebraic manipulation of (13) the following may be obtained as explicitly shown in [21],

$$\left(x - \frac{\mu^2 d}{1-\mu^2}\right)^2 + y^2 = \left(\frac{\mu d}{1-\mu^2}\right)^2 \tag{14}$$

From the form of (14), a circle in the Cartesian plane is defined and shown in Fig. 1, where the distances,

$$\overline{OE} = \frac{d\mu^2}{1-\mu^2}, \quad \overline{EI} = \frac{d\mu}{1-\mu^2}. \tag{15}$$

The Apollonius circle shown in Fig. 1 provides the locus of all possible interceptions of the evader by the pursuer under the assumption that the pursuer is not range-limited. From the law of sines the optimal strategy for the pursuer, provided the evader's choice of heading is obtained:

$$\frac{\overline{PI}}{\sin(\pi - \psi_E)} = \frac{\overline{EI}}{\sin\psi_P} = \frac{\mu\overline{PI}}{\sin\psi_P} \tag{16}$$

solving for the optimal heading ψ_P:

$$\psi_P = \sin^{-1}(\mu \sin\psi_E) \tag{17}$$

From the law of cosines:

$$\overline{PI}^2 = \mu^2\overline{PI}^2 + d^2 - 2d\mu\overline{PI}\cos(\pi - \psi_E) \tag{18}$$

Note that $\cos(\pi - \psi_E) = -\cos(\psi_E)$.

$$\overline{PI}^2 = \mu^2\overline{PI}^2 + d^2 + 2d\mu\overline{PI}\cos\psi_E \tag{19}$$

Using the quadratic equation, $\overline{PI}$ may be obtained,

$$\overline{PI} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{20}$$

where

$$a = (1-\mu^2),\ b = -2d\mu\cos\psi_E,\ c = -d^2.$$

The positive case is of interest because the distance $\overline{PI}$ is a strictly positive quantity. Through algebraic manipulation one obtains

$$\overline{PI} = \frac{d}{1-\mu^2}\left(\mu\cos\psi_E + \sqrt{1-\mu^2\sin^2\psi_E}\right). \tag{21}$$

Using (10), $\overline{EI}$ can be obtained.

In summary, for a pursuer with unlimited range that captures an evader using point-capture, the following are the pursuer's optimal heading and range to capture, provided the evader's choice of heading, ψ_E, and speed ratio μ.

$$\begin{aligned}\psi_P &= \sin^{-1}(\mu\sin\psi_E) \\ \overline{PI} &= \frac{d}{1-\mu^2}\left(\mu\cos\psi_E + \sqrt{1-\mu^2\sin^2\psi_E}\right)\end{aligned}$$

IV. Range-Limited Point Capture

Shifting the Apollonius circle as defined in (14) such that the Pursuer is located at the origin, $P = (0,0)$ and the evader is located at $E = (d, 0)$ means that the resulting Apollonius circle is shifted along the x-axis and is

$$\left(x - \left(d + \frac{\mu^2 d}{1-\mu^2}\right)\right)^2 + y^2 = \left(\frac{\mu d}{1-\mu^2}\right)^2. \tag{22}$$

The pursuer may reach any point in the circle

$$x^2 + y^2 = R^2. \tag{23}$$

Define the following regions of Cartesian space:

$$\mathscr{E} \triangleq \left\{(x,y)\middle|\left(x - \left(d + \tfrac{\mu^2 d}{1-\mu^2}\right)\right)^2 + y^2 \le \left(\tfrac{\mu d}{1-\mu^2}\right)^2\right\}$$

$$\mathscr{P} \triangleq \left\{(x,y)\middle| x^2 + y^2 \le R\right\}$$

The region $\mathscr{E}$ defines the reachable region of the evader until it is captured by a pursuer without limited range. The region $\mathscr{P}$ defines the reachable region of the pursuer. Three cases of interest exist:

1) $\mathscr{P} \cap \mathscr{E} = \varnothing$. The reachable region of the pursuer does not intersect the Apollonius circle. Capture is not possible, independent of the evader's choice of headings.
2) $\mathscr{P} \cup \mathscr{E} = \mathscr{P}$. The reachable region of the pursuer completely envelops the Apollonius circle. Capture is ensured and dictated by the Apollonius circle.
3) $\mathscr{P} \cap \mathscr{E} \neq \varnothing$ and $\mathscr{P} \cup \mathscr{E} \neq \mathscr{P}$. There exists an interval of headings for which the evader can escape capture because the pursuer is range-limited.

A. Evader Always Escapes

The evader always escapes when the reachable region of the pursuer does not intersect the Apollonius circle between the pursuer and evader. Such a case is shown in Fig. 2. This occurs when

$$R < d + \frac{\mu^2 d}{1-\mu^2} - \frac{\mu d}{1-\mu^2}. \tag{24}$$

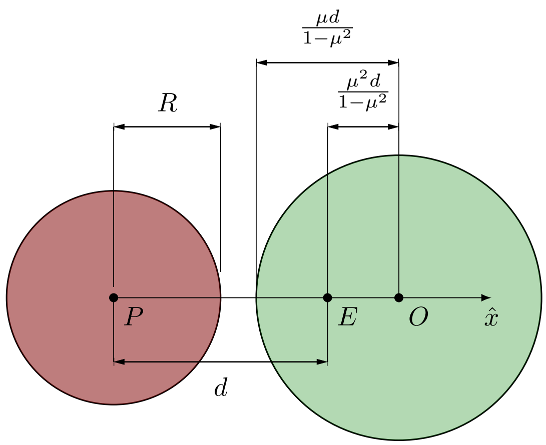

Fig. 2. The pursuer's range is too small and it is unable to reach the evader, independent of the evader's choice of heading.

Simplifying, (24) can be rewritten as

$$R < \frac{d}{1+\mu}. \tag{25}$$

In this case, the pursuer's range is too small and it is unable to capture the evader. In summary, when the pursuer is unable to reach the Apollonius circle, $\mathscr{P} \cap \mathscr{E} = \varnothing$, the pursuer's strategy and time to reach the interception point are as follows:

$$\boxed{\begin{aligned} \psi_P &= \text{undefined} \\ \overline{PI} &= \infty \end{aligned}}$$

B. Evader Capture Guaranteed

In the event that the pursuer's range is so large that it completely envelops the Apollonius circle, capture is ensured and dictated by the Apollonius circle. This case is shown in Fig. 3. This scenario occurs when the following is true

$$R \geq d + \frac{\mu^2 d}{1-\mu^2} + \frac{\mu d}{1-\mu^2}. \tag{26}$$

Simplifying, (26) can be rewritten as

$$R \geq \frac{d}{1-\mu}. \tag{27}$$

In summary, when the pursuer's reachable region completely encompasses the Apollonius circle, $\mathscr{P} \cup \mathscr{E} = \mathscr{P}$, and therefore the optimal heading for the pursuer and the range until interception is provided by the Apollonius geometry:

$$\boxed{\begin{aligned} \psi_P &= \sin^{-1}(\mu \sin \psi_E) \\ \overline{PI} &= \frac{d}{1-\mu^2}\left(\mu \cos \psi_E + \sqrt{1-\mu^2 \sin^2 \psi_E}\right) \end{aligned}}$$

C. Limited Capture Region

By (25) and (27), it follows that there exist an interval of headings for which the evader can escape point-capture by a range-limited pursuer when the following holds,

$$\frac{d}{1+\mu} \leq R < \frac{d}{1-\mu}. \tag{28}$$

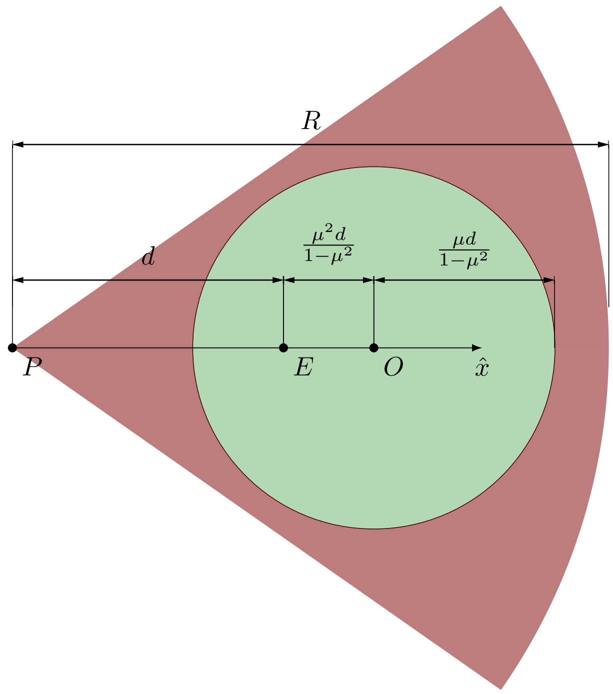

Fig. 3. The pursuer's range is so large that it completely envelops the Apollonius circle and it is able to capture the evader, independent of the evader's choice of heading.

This is demonstrated in Fig. 4. The location of the interception points are obtained by substituting (23) into (14),

$$\left(x - \left(d + \frac{\mu^2 d}{1-\mu^2}\right)\right)^2 + (R^2 - x^2) = \left(\frac{\mu d}{1-\mu^2}\right)^2. \tag{29}$$

Solving (29) for x, the following is obtained,

$$x_I = \frac{d^2(1-\mu^2) + 2\mu^2 d^2 + R^2(1-\mu^2)}{2d(1+\mu^2)}. \tag{30}$$

Substituting (30) into (23), the y-coordinate for the interception points are:

$$y_I = \pm\sqrt{R^2 - \left(\frac{(d^2+R^2)(1-\mu^2)+2\mu^2 d^2}{2d(1+\mu^2)}\right)^2} \tag{31}$$

The headings for which the evader escapes the range-limited pursuer are

$$\psi_E \in \left[0, \operatorname{atan}\left(\frac{y_{I,+}}{x_I}\right)\right) \cup \left(\operatorname{atan}\left(\frac{y_{I,-}}{x_I}\right), 2\pi\right]. \tag{32}$$

Alternatively, one may obtain the safe range of evader headings via the Law of Cosines. Consider, for example, the triangle $\triangle PI_1E$; using $\pi - \psi_E$ as angle, the Law of Cosines is

$$\overline{PI}^2 = d^2 + \overline{EI}^2 - 2d\overline{EI}\cos(\pi - \psi_E). \tag{33}$$

Substituting in $\overline{PI} = R$, $\overline{EI} = \mu R$, and solving for ψ_E gives

$$\psi_{E,\text{ crit}} = \cos^{-1}\left(\frac{(1-\mu^2)R - d^2}{2d\mu R}\right), \tag{34}$$

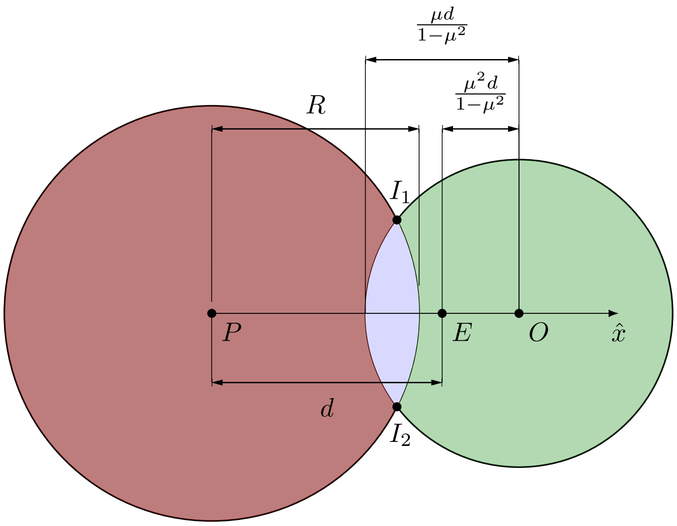

Fig. 4. The pursuer's range intersects the Apollonius circle and there exist an interval of headings that the evader can take that ensure that it can avoid point-capture by the pursuer (green).

and thus evader headings which satisfy the following are safe:

$$\cos\psi_E > \cos\psi_{E,\text{ crit}}. \tag{35}$$

This is equivalent to (32).

In summary, provided the evader's choice of heading, the pursuer may or may not be able to capture the evader. The following describes the optimal strategy for the pursuer.

$$\psi_P = \begin{cases} \sin^{-1}(\mu\sin\psi_E) & \psi_E \notin \text{Eq.}(32) \\ \text{undefined} & \psi_E \in \text{Eq.}(32) \end{cases}$$

$$\overline{PI} = \begin{cases} \frac{d}{1-\mu^2}\left(\mu\cos\psi_E + \sqrt{1-\mu^2\sin^2\psi_E}\right) & \psi_E \notin (32) \\ \infty & \psi_E \in (32) \end{cases}$$

V. Non-Zero Capture Radius

Consider a faster pursuer that is endowed with a non-zero capture radius, ρ. The pursuer is able to capture the evader when the evader is within a specified capture radius, $\rho > 0$. Capture is defined when the state reaches the capture set $\mathscr{C}_{rc} \triangleq \{\mathbf{x} | (x_P - x_E)^2 + (y_P - y_E)^2 = \rho^2\}$. As shown earlier, since both the pursuer and evader exhibit simple motion (both agents may turn instantaneously), the costates under optimal play are constant by (7) and the resulting optimal pursuer and evader trajectories are straight lines. The outcome of such a trajectory under the assumption that the pursuer is not range-limited can be seen in Fig. 5.

The objective of the non-zero capture radius problem is to obtain the heading for the pursuer that captures the constant bearing evader in minimum time.

A. Pursuer Distance

Provided the evader's choice of heading, the distance that the pursuer travels in order to capture the slower evader in minimum time is obtained. Capture is assumed to occur when

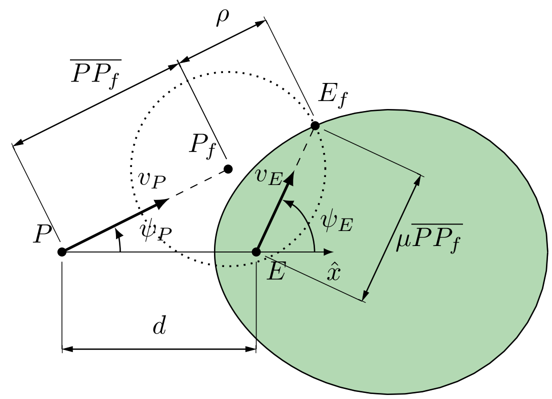

Fig. 5. When the pursuer is endowed with a non-zero capture radius the resulting locus of points dictating the capture of a slower evader by a faster pursuer is a quartic plane curve also referred to as a Cartesian oval.

the state of the systems is in the capture set $\mathbf{x}(t_f) \in \mathscr{C}_{rc}$. Consider the triangle: $\triangle PE_fE$ in Fig. 5. From the law of cosines:

$$(\overline{PP_f} + \rho)^2 = d^2 + \mu^2\overline{PP_f}^2 - 2d\mu\overline{PP_f}\cos(\pi - \psi_E) \tag{36}$$

Simplifying and factoring (36) in terms of $\overline{PP_f}$, the following is obtained:

$$(1-\mu^2)\overline{PP_f}^2 + (2\rho - 2d\mu\cos\psi_E)\overline{PP_f} + \rho^2 - d^2 = 0. \tag{37}$$

A quadratic equation in $\overline{PP_f}$ is obtained thereby obtaining the distance that the pursuer travels prior to the capture of the evader, provided the evader's choice of heading ψ_E, the speed ratio of the two agents, μ, and the separation of the two agents, d.

$$\overline{PP_f} = \frac{d\mu\cos\psi_E - \rho \pm \sqrt{(\rho - d\mu\cos\psi_E)^2 - (1-\mu^2)(\rho^2 - d^2)}}{(1-\mu^2)} \tag{38}$$

It must be the case that $d > \rho$, otherwise the evader is already within the capture radius of the pursuer. Therefore, the term inside the radical is always positive (since $\mu < 1$). Moreover, the entire square root term is larger than the $d\mu\cos\psi_E - \rho$ term, and thus, for $\overline{PP_f}$ to be positive, we only consider the positive version of this expression:

$$\overline{PP_f} = \frac{d\mu\cos\psi_E - \rho + \sqrt{(\rho - d\mu\cos\psi_E)^2 - (1-\mu^2)(\rho^2 - d^2)}}{(1-\mu^2)}. \tag{39}$$

B. Pursuer Heading

The heading that the pursuer takes in order to capture the evader in minimum time is obtained using the law of sines.

$$\frac{\overline{PP_f} + \rho}{\sin(\pi - \psi_E)} = \frac{\overline{EE_f}}{\psi_P} = \frac{\mu\overline{PP_f}}{\sin\psi_P}. \tag{40}$$

Solving (40) for ψ_P as a function of ψ_E,

$$\psi_P = \sin^{-1}\left(\frac{\mu\overline{PP_f}\sin\psi_E}{\overline{PP_f} + \rho}\right) \tag{41}$$

where $\overline{PP_f}$ is obtained in (38), ρ is the capture radius, μ is the speed ratio, and ψ_E is the evader's choice of heading.

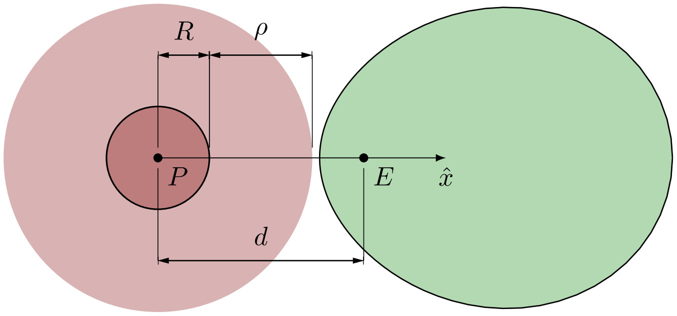

Fig. 6. The range limited pursuer has an extended reachable region due to its capture radius, ρ. This region does not intersect the Cartesian oval between the pursuer and evader

In summary, provided the evader's choice of heading, when the pursuer has a non-zero capture radius, the heading and range of the pursuer can be obtained in closed form.

$$\overline{PP_f} = \frac{d\mu\cos\psi_E - \rho + \sqrt{(\rho - d\mu\cos\psi_E)^2 - (1-\mu^2)(\rho^2 - d^2)}}{(1-\mu^2)}$$
$$\psi_P = \sin^{-1}\left(\frac{\mu\overline{PP_f}\sin\psi_E}{\overline{PP_f}+\rho}\right)$$

From this result the Cartesian oval that describes the safe-region of the evader prior to capture by the pursuer can be obtained. The distance the evader travels is proportional to the distance traveled by the pursuer,

$$\overline{EE_f} = \mu\overline{PP_f}. \tag{42}$$

Substitution of (38) into (42) yields the polar equation for the Cartesian oval,

$$\overline{EE_f}(\psi_E;\mu,d) = \frac{\mu}{1-\mu^2}(d\mu\cos\psi_E - \rho + \sqrt{(\rho - d\mu\cos\psi_E)^2 - (1-\mu^2)(\rho^2-d^2)}) \tag{43}$$

VI. Range-Limited Non-Zero Capture Radius

Define the following regions in Cartesian space:

$$\mathcal{E}_\rho \triangleq \left\{E^\dagger = (x,y) \mid \overline{EE^\dagger} \le \overline{EE_f}(\angle EE^\dagger)\right\}$$
$$\mathcal{P}_\rho \triangleq \left\{(x,y) \mid x^2 + y^2 \le R + \rho\right\},$$

where $\overline{EE_f}$ is given by (43). As in Section III, there are three possibilities for the outcome on the scenario which depends on the existence (or non-existence) of overlap between these two regions.

A. Evader Escape Guaranteed

Assume that the evader heads directly toward the pursuer, we aim to find the range, R, such that

$$d - \mu R > R + \rho \tag{44}$$

solving for R, the inequality is

$$R < \frac{d-\rho}{1+\mu} \tag{45}$$

Such a case is illustrated in Fig. 6.

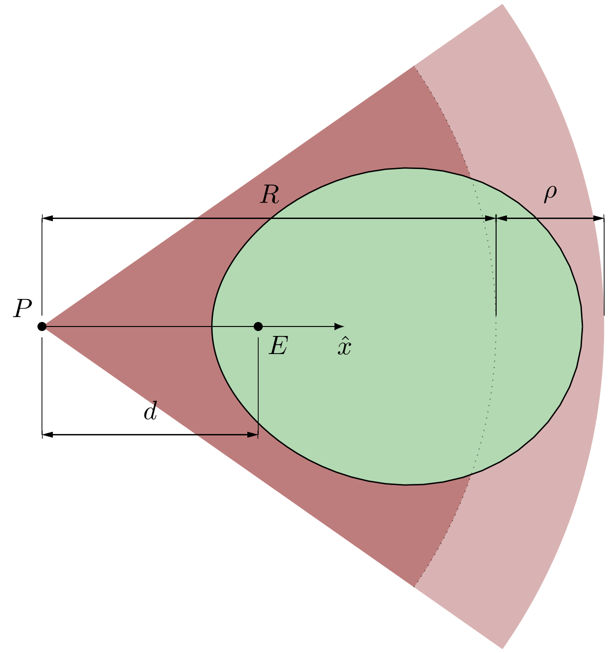

Fig. 7. The combination of the pursuer's range and capture radius are such that its total reachable region covers the Cartesian oval representing the evader's reachable region. Given ψ_E, the pursuer can guarantee the capture of the evader.

In summary, when the pursuer is unable to reach the evader's reachable region, $\mathcal{P}_\rho \cap \mathcal{E}_\rho = \varnothing$, the pursuer's strategy and time to reach the interception point are as follows:

$$\psi_P = \text{undefined}$$
$$\overline{PP_f} = \infty$$

B. Evader Capture Guaranteed

Similarly, capture of the evader is guaranteed if capture occurs when the evader runs directly away:

$$d + \mu R \le R + \rho \tag{46}$$

Solving for R, the inequality is

$$R \ge \frac{d-\rho}{1-\mu} \tag{47}$$

In this case, the pursuer's reachable region completely overlaps the evader's reachable region, $\mathcal{E}_\rho \subset \mathcal{P}_\rho$, as seen in Fig. 7. Therefore, the optimal heading for the pursuer and range until interception, given the evader's heading, is provided by the Cartesian oval geometry:

$$\psi_P = \sin^{-1}\left(\frac{\mu\overline{PP_f}\sin\psi_E}{\overline{PP_f}+\rho}\right)$$
$$\overline{PP_f} = \frac{d\mu\cos\psi_E - \rho + \sqrt{(\rho - d\mu\cos\psi_E)^2 - (1-\mu^2)(\rho^2 - d^2)}}{(1-\mu^2)}$$

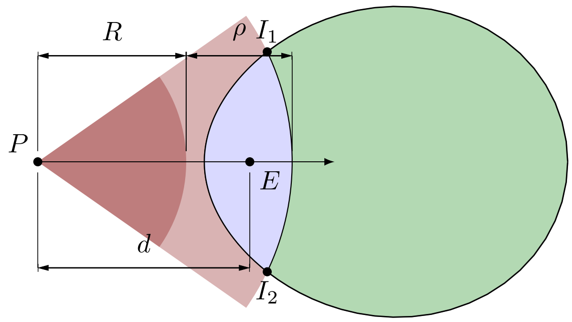

Fig. 8. The pursuer's reachable region intersects the evader's Cartesian oval and thus there is an interval of evader headings for which capture is possible and an interval for escape is guaranteed.

C. Limited Evader Capture

The Cartesian oval that represents the capture of the evader by the pursuer with $\rho > 0$ is of interest. Define $\beta = \frac{v_P}{v_E} > 1$. The Cartesian Oval is

$$\sqrt{x^2 + y^2} = \beta\sqrt{(x - x_E)^2 + y^2} + \rho \tag{48}$$

$$x = \frac{2\rho R - \rho^2 + \beta^2 x_E^2 - R^2(1-\beta^2)}{2\beta^2 x_E} \tag{49}$$

The critical evader headings (i.e., the limiting case of capture) which lead to the evader's capture at the points I_2 or I_1 may be obtained by substituting $\overline{PP_f} = R$ into (36) and solving for ψ_E:

$$\psi_{E,\text{ crit}} = \cos^{-1}\left(\frac{(R+\rho) - d^2 - \mu^2 R^2}{2d\mu R}\right). \tag{50}$$

Thus, the headings for which the evader escapes the range-limited, non-zero capture radius pursuer satisfy

$$\cos\psi_E > \cos\psi_{E,\text{ crit}}. \tag{51}$$

If the evader's heading is such that $\cos\psi_E \leq \cos\psi_{E,\text{ crit}}$ then the corresponding pursuer heading which leads to capture is given by (41).

In summary, provided the evader's choice of heading, the pursuer may or may not be able to capture the evader. The following describes the optimal strategy for the pursuer

$$\psi_P = \begin{cases} \sin^{-1}\left(\frac{\mu\overline{PP_f}\sin\psi_E}{\overline{PP_f}+\rho}\right) & \text{if } \cos\psi_E \leq \cos\psi_{E,\text{ crit}}, \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$\overline{PP_f} = \begin{cases} (39) & \text{if } \cos\psi_E \leq \cos\psi_{E,\text{ crit}}, \\ \infty & \text{otherwise} \end{cases}$$

VII. Conclusions

Much of the pursuit-evasion literature has focused on the case in which the pursuer is unconstrained - in particular, the pursuer is assumed to have infinite range. In reality, whether due to limited on-board fuel or finite communication ranges, a pursuer is more likely to have a limit imposed on its range. In the case that the pursuer moves in a straight line (as in this work), this becomes analogous to scenarios with a fixed time horizon. This work is based on a fairly restrictive assumption that the pursuer has knowledge of the evader's heading. Nevertheless, this analysis provides worst-case guarantees (for the evader) for scenarios in which the evader's heading is not known to the pursuer. Given the evader's heading, the optimal pursuer heading and travel distance leading to capture are provided for both the point capture and non-zero capture radius cases. These results provide a basis for considering more complex scenarios such as two range-limited pursuers versus a single evader.

Acknowledgment

This paper is based on work performed at the Air Force Research Laboratory (AFRL) *Control Science Center*. Distribution Unlimited. 28 Apr 2023. Case #AFRL-2023-2058.

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