Engagement Zone Defense of a Non-Maneuvering Evader

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***Abstract*— This paper considers a three agent scenario consisting of a pursuer, evader, and a defender. The pursuer's objective is to capture the non-maneuvering evader in minimum time while a defender aims at maximize contact with the pursuer by keeping the pursuer inside his circular engagement zone for as long as possible; the pursuer is considered to be faster than both the evader and the defender. Using optimal control theory, the optimal control for the defender that maximizes contact with the pursuer is posed and solved. In the event that the evader is captured by the pursuer before the pursuer escapes the engagement zone of the defender, some suboptimal strategies of the defender provide equivalent contact time. A derivation of defender's headings that maximize contact is presented along with examples that highlight the importance of the initial conditions of the engagement scenario.**

I. Introduction

Directed energy (DE) warfare is a means of using the electromagnetic spectrum (EMS) to achieve one of many military objectives including, but not limited to, the protection of friendly facilities. Rather than providing kinetic defense (KD) by capturing a target vehicle, DE devices expose a desired target over time. Furthermore, electronic warfare (EW) applications in support of homeland defense are vital to deter, detect, prevent, and defeat external threats such as ballistic missiles, aircraft (manned and unmanned), maritime vessels, land threats, hostile space systems, domestic/international terrorism, and cyberspace threats [1]. In this paper, we consider the defense of a non-maneuvering evader against an interceptor.

In video-games the engagement zone (EZ) as described in [1] is referred to as the area of effect (AoE) [2], [3]. The basic game mechanics of an AoE are as follows: there exists a region (usually circular) which surrounds a specific player; whenever other players are located inside that boundary they are effected by some predetermined effect which could be friendly or adversarial in nature. This concept is akin to electronic warfare, wherein the EMS is used to potentially observe or inhibit the function of an adversary or friendly asset.

Differential games where the objective of one of the agents is to maximize the observation of an evader who aims at minimizing his observation are commonly refereed to as "Surveillance Evasion Differential Games." Early work in Surveillance Evasion Differential Games (SEDGs) was presented by Dobbie [4]. In his paper, Dobbie considered a game of kind, determining if the pursuer is able to keep the evader inside his detection region or not. Others extended the work of Dobbie to consider the game of degree for the SEDG [5]–[7]. In each of the works [4]–[7] the pursuer was assumed to be faster than the evader; but, turn restrictions for the pursuer were assumed, making the problem very relatable to the game of two-cars [8], [9]. Fuchs and Metcalf also presented a SEDG in which a mobile radar attempts to gather sufficient information to identify a mobile target in minimum time; simultaneously, the target attempts to maneuver in such a way to maximize the identification time [10]. In this work, the pursuer and evader from the SEDG is called the "defender" and "pursuer" respectively; also, the defender is assumed to be slower than the pursuer.

Other observation problems related to the SEDG are found in [11]–[14]. LaValle et al. posed and solved a surveillance evasion optimization problem (SEOP) consisting of an observer and a target in a field of occluding obstacles [11]. Cartee et al. further investigated the SEOP under uncertainty where the observer implements a fixed trajectory [12]. Ly and Tsai also considered the SEOP consisting of multiple observers and targets allowing for the observer(s) to vary their course [13]. Garnett and Flenner presented an SEOP between a faster intelligence, surveillance and reconnaissance (ISR) platform and slower aerial targets [14]. Because of the model complexity a nonlinear program (NLP) solver was used to solve the SEOP. He et al. proposed an air-to-ground SEOP wherein a helicopter, with limited information, tracks multiple ground targets restricted to a road network [15].

In each of the SEOPs and SEDGs above, the observer is assumed to be faster than the target. Works which have considered the pursuing agent to be slower than the evading agent are found in [16], [17]. Breakwell presented a pursuit-evasion differential game wherein a slower pursuer aims at minimizing the miss distance to the evader in the event that the evader can not be captured. Breakwell, also considered a nonzero capture radius for the slower pursuer; and terminated the game when the pursuer captured the slower evader [16]. Weintraub et al. extended the work of Breakwell by posing an optimization problem which maximizes the time a target remains (continuously) inside an observer's circular detection region for as long as possible [17].

The defense of an evader from an adversarial pursuer has been considered in the past [18]–[22]. This scenario is commonly referred to as "Active Target Defense." Boyell

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proposed the defense of a naval vessel by means of a defensive torpedo against an incoming threat [18], [19]. Pachter, Garcia, and Casbeer have studied active target defense by kinetic means in great detail [20]–[22].

In this paper, the consideration of directed energy as a means of defending a slow, non-maneuverable evader is considered. In the engagement a faster pursuer aims at intercepting the non-maneuvering evader in minimum time. In an effort to aid the evader, a teamed defender equipped with a directed energy device is considered. Rather than focus on the technology or the more critical aspects of the directed energy device, it is assumed that the efficacy of the defensive device is increased by maximizing the time that the pursuer is in an effective engagement zone (EZ) of fixed range about the defender; and by this assumption, the objective of the defender is to maximally expose the pursuer prior to capture of the evader.

II. Optimal Control Problem

Consider a three-agent engagement scenario comprised of a fast pursuer (P) which is engaged on a slow non-maneuvering evader (E) – the evader moves on a straight-line trajectory. In order to aid in the defense of the evader, a defender (D) is considered which aims at keeping the pursuer in its engagement zone (EZ) for as long as possible. For this preliminary work, a circular EZ with radius R_D is considered. Rather than focus on the mechanism of the defender, this paper considers the question: How should the defender act so as to keep the pursuer in his circular EZ for as long as possible? It is assumed that the pursuer's strategy is one of min-time capture of the evader and as a result, its strategy is unaffected by the presence of the defender.

Consider the simple motion model for all agents where the velocity of the pursuer, defender, and evader are v_P, v_D, and v_E respectively. The complete state of the engagement is $\mathbf{x} = [x_P, y_P, x_E, y_E, x_D, y_D] \in \mathbb{R}^6$, where (x_P, y_P), (x_E, y_E), and (x_D, y_D) are the positions of P, E, and D respectively. Also define ψ_P, ψ_E, and ψ_D as the instantaneous heading of P, E, and D respectively. Motion is to be restricted to the 2-dimensional plane, which is a common assumption in the pursuit-evasion scenarios of Isaacs [8]. The equations of motion for the three-agent scenario are

$$\begin{aligned} \dot{x}_P &= v_P \cos\psi_P & \dot{x}_E &= v_E \cos\psi_E & \dot{x}_D &= v_D \cos\psi_D \\ \dot{y}_P &= v_P \sin\psi_P & \dot{y}_E &= v_E \sin\psi_E & \dot{y}_D &= v_D \sin\psi_D. \end{aligned} \tag{1}$$

The speed ratio between the pursuer and evader is defined as $\mu = v_E/v_P$. Similarly, the speed ratio between the pursuer and the defender is defined as $\alpha = v_D/v_P$. Since the pursuer is faster than both the defender and the evader the domain of the speed ratios is μ, $\alpha \in (0, 1)$. Without loss of generality, consider the nondimensionalization of time so that the pursuer's speed is unity. Utilizing the speed ratio parameter, the equations of motion in (1) may be simplified to be

$$\begin{aligned} \dot{x}_P &= \cos\psi_P & \dot{x}_E &= \mu \cos\psi_E & \dot{x}_D &= \alpha \cos\psi_D \\ \dot{y}_P &= \sin\psi_P & \dot{y}_E &= \mu \sin\psi_E & \dot{y}_D &= \alpha \sin\psi_D \end{aligned} \tag{2}$$

The non-maneuvering evader is on a fixed straight-line course and its position and heading are known by the pursuer. The pursuer, knowing the state of the evader, wishes to select a heading which intercepts E in minimum time. Capture is effected when P and E are coincident (i.e., point-capture). During this pursuit, the defender has a circular EZ with radius R_D, and it desires to keep the pursuer inside its EZ for the maximum possible continuous time; that is, that P remains inside the EZ without interruption.

The initial conditions for the scenario are that P and E are located in arbitrary locations in the 2-dimensional plane; while D is located a distance R_D from P. At time zero (t_0), the initial state $\mathbf{x}(t_0) \equiv \mathbf{x}_0 \in \mathscr{I}$, where

$$\mathscr{I} = \{\mathbf{x} | \sqrt{(x_P - x_D)^2 + (y_P - y_D)^2} - R_D = 0\}.$$

The termination set which represents the point-capture of the evader by the pursuer is defined as

$$\mathscr{C}_A = \{\mathbf{x} | (x_P - x_E)^2 + (y_P - y_E)^2 = 0\}. \tag{3}$$

The instant where the state satisfy (3) is defined as t_{go}; also called the "time-to-go". The termination set which represents the escape of the pursuer from the defender is

$$\mathscr{C}_B = \{\mathbf{x} | (x_P - x_D)^2 + (y_P - y_D)^2 - R_D^2 > 0\}. \tag{4}$$

The instant where the state satisfies (4) is defined as t_{exp}; also called the "exposure time." The termination set of the entire scenario is $\mathscr{C}_A$, that is, the pursuer captures the evader regardless if the pursuer has escaped the EZ of the defender prior to capturing the evader or if the pursuer has captured the evader before escpaing the defender's EZ. One objective of this paper is to analyze the optimal strategy of the defender when t_{exp} is either less than, greater than, or equal to t_{go}. Also presented are the conditions when $t_{\text{exp}} = 0$, that is, that the defender can not expose the pursuer at all no matter the strategy of the defender.

The objective of the pursuer is to capture the evader in minimum time – to make t_{go} a minimum. The objective cost functional of the pursuer is

$$J_A(\mathbf{x}_0; \psi_P(\cdot)) = \int_0^{t_{\text{go}}} 1 \, \mathrm{d}t = t_{\text{go}} \tag{5}$$

The optimal time-to-go is $t_{\text{go}}^* = \min J_A$ subject to the termination set in (3) – the pursuer and evader are collocated at final time. The goal for the pursuer is to find his optimal heading which minimizes the objective cost functional in (5), namely

$$\psi_P^*(t) = \underset{\psi_P}{\operatorname{argmin}} \, J_A \tag{6}$$

The objective of the defender is to keep the pursuer inside his EZ for as long as possible. Since the pursuer is faster than the defender, his escape is guaranteed for a finite EZ range, R_D. The objective cost function of the defender is

$$J_B\left(\mathbf{x}_0; \psi_D(\cdot)\right) = \int_0^{t_f} -1 \, \mathrm{d}t = -t_f \tag{7}$$

where the final time, $t_f = \min(t_{\text{exp}}, t_{\text{go}})$. This optimization problem ends when the states of the scenario reaches $\mathscr{C} = \mathscr{C}_A \cup \mathscr{C}_B$.

The optimal exposure time is $t^*_{\text{exp}} = \min J_B$ subject to the termination set in (4) – the pursuer is no longer contained inside the EZ of the defender. The goal is to find the optimal defender's heading which minimizes the objective cost functional in (7), namely

$$\psi^*_D(t) = \underset{\psi_D}{\text{argmin}}\ J_B \tag{8}$$

Two optimization problems are formulated and solved, and their interaction is investigated in this paper. The costate vectors $\mathbf{p}_A = [p_{x_P,A}\ \ p_{y_P,A}\ \ p_{x_E,A}\ \ p_{y_E,A}]$ and $\mathbf{p}_B = [p_{x_P,B}\ p_{y_P,B}\ p_{x_D,B}\ p_{y_D,B}]$ are introduced in order to formulate the Hamiltonians for solving the two optimization problems (A and B) as defined by the minimization of the cost functionals in (5) and (7). Using the optimal control theory, the Hamiltonian for the minimization described in (3) and (5) is the following:

$$\begin{aligned}\mathscr{H}_A = p_{x_P}\cos\psi_P + p_{y_P}\sin\psi_P \\ + p_{x_E}\mu\cos\psi_E + p_{y_E}\mu\sin\psi_E\end{aligned} \tag{9}$$

The Hamiltonian for the minimization described in (4) and (7) is the following:

$$\begin{aligned}\mathscr{H}_B = p_{x_P}\cos\psi_P + p_{y_P}\sin\psi_P \\ + p_{x_D}\alpha\cos\psi_D + p_{y_D}\alpha\sin\psi_D\end{aligned} \tag{10}$$

A. Necessary Conditions for Optimality

The procedure for solving the optimal strategies of the pursuer and the defender are to first formulate and solve for the min-time capture of the evader by the pursuer – to solve the optimization problem as described in (3), (5), (6) and (9). Then, using the solution to the optimization problem A, the optimal strategy for the defender is then posed and solved as described by (4), (7), (8) and (10).

Using the first-order optimality conditions, conclusions about the optimal behavior of the defender and the pursuer can be drawn. Using the Hamiltonians in (9) and (10), the necessary conditions for optimality are found using the following partial derivatives:

$$\dot{\mathbf{x}}^*(t) = \tfrac{\partial\mathscr{H}(\mathbf{x}^*(t),\mathbf{p}^*(t),\mathbf{u}^*(t),t)}{\partial\mathbf{p}^*(t)} \tag{11}$$

$$\dot{\mathbf{p}}^*(t) = -\tfrac{\partial\mathscr{H}(\mathbf{x}^*(t),\mathbf{p}(t)^*,\mathbf{u}^*(t),t)}{\partial\mathbf{x}^*(t)} \tag{12}$$

$$\mathbf{0} = \tfrac{\partial\mathscr{H}(\mathbf{x}^*(t),\mathbf{p}^*(t),\mathbf{u}^*(t),t)}{\partial\mathbf{u}^*(t)} \tag{13}$$

$\mathscr{H}_A(t_{\text{go}}) = \mathscr{H}_B(t_{\text{exp}}) = 0$ and the superscript, $*$, represents optimality. Also, the control, $\mathbf{u}(t)$, for the pursuer or defender (depending upon the problem being solved) are defined as $\mathbf{u}(t) = \psi_P(t)$ for problem A and $\mathbf{u}(t) = \psi_D(t)$ for problem B.

B. Optimal Control Problem, A - Pursuer Strategy

Lemma 1. *The optimal strategy for the pursuer is a straight-line constant-heading strategy.*

Proof. Evaluating (12) using the Hamiltonian, $\mathscr{H}_A$ from (9) the costate dynamics are as follows:

$$\dot{p}^*_{x_P,A} = \dot{p}^*_{y_P,A} = \dot{p}^*_{x_E,A} = \dot{p}^*_{y_E,A} = 0 \tag{14}$$

Further, evaluating the partial of $\mathscr{H}_A$ with respect to the pursuer's control ψ_P in (13),

$$0 = -p^*_{x_P,A}\sin\psi^*_P + p^*_{y_P,A}\cos\psi^*_P. \tag{15}$$

Rearranging and squaring each side of (15), the following is obtained:

$$p^{*2}_{x_P,A}\sin^2\psi^*_P = p^{*2}_{y_P,A}\cos^2\psi^*_P = p^{*2}_{y_P,A}(1-\sin^2\psi^*_P) \tag{16}$$

solving for ψ^*_P,

$$\psi^*_P = \arcsin\left(p^*_{y_P,A}/\sqrt{p^{*2}_{x_P,A} + p^{*2}_{y_P,A}}\right) \tag{17}$$

Since the costates are constant, the heading of the pursuer is constant under optimal play. ■

Thus, the use of Apollonius circle is a useful tool for solving for the min-time interception of the evader by the pursuer [23].

Lemma 2. *The optimal heading for the pursuer which captures the non-maneuvering evader in minimum time is $\psi^*_P = \sin^{-1}(\mu\sin(\psi_E - \theta_E)) + \theta_E$ where μ is the speed ratio between the evader and the pursuer, ψ_E is the heading of the evader, and θ_E is the angle from the pursuer to the evader relative to the x-axis in the global frame.*

Proof. From necessary conditions for optimality, the optimal strategy for the pursuer is a straight light trajectory. Moreover, the evader is non-maneuvering and therefore, the geometry of Apollonius provides the optimal strategy for the pursuer to capture the slower non-maneuvering evader in minimum time [8].

In Figure 1, the Apollonius circle is shown in magenta, where the origin is located at O. The distance $d = |\overline{PE}|$, the speed ratio $\mu = \frac{v_E}{v_P}$, the position of P and E are used to construct the Apollonius circle which defines the locus of min-time interceptions possible by the pursuer because the evader is non-maneuvering. Along the vector $\vec{PE}$ the geometry defined by Apollonius: $\overline{OE} = \frac{\mu^2 d}{1-\mu^2}$ and $R_{apol} = \overline{OI} = \frac{\mu d}{1-\mu^2}$.

Evaluating the distance normal to $\overline{PO}$ which locates point I from E and P, the following can be obtained.

$$\overline{EI}\sin(\psi_E - \theta_E) = \overline{PI}\sin(\psi^*_P - \theta_E) \tag{18}$$

Recall, the speed ratio μ defines the relationship between $\overline{EI}$ and $\overline{PI}$ to be: $\overline{EI} = \mu\overline{PI}$. Substitution into (18) the following is obtained:

$$\mu\overline{PI}\sin(\psi_E - \theta_E) = \overline{PI}\sin(\psi^*_P - \theta_E) \tag{19}$$

Solving for ψ^*_P:

$$\psi^*_P = \sin^{-1}(\mu\sin(\psi_E - \theta_E)) + \theta_E \quad ■ \tag{20}$$

Lemma 3. *Using the geometry provided by Apollonius' circle, the time-to-go can be obtained as*

$$t_{go} = d(\sigma_1 + \sqrt{\sigma_1^2 + \sigma_2}) \tag{21}$$

where $\sigma_1 = \mu\cos(\psi_E - \theta_E)/(1-\mu^2)$ and $\sigma_2 = 1/(1-\mu^2)$.

Proof. Because the velocity of the pursuer has been normalized to 1, the time-to-go in seconds is the same as the length $\overline{PI}$. This means, the length $\overline{PI}$ provides the time-to-go, t_{go}. Consider $\triangle IEO$; by the law of cosines,

$$\overline{OI}^2 = \overline{EO}^2 + \overline{EI}^2 - 2\overline{EO}\,\overline{EI}\cos(\psi_E - \theta_E) \tag{22}$$

Define the distance between the pursuer and evader as d and the evader-pursuer ratio is $v_E/v_P = \mu$, where $\mu < 1$. From the Apollonius circle, $\overline{EO} = \mu^2 d/(1-\mu^2)$, $\overline{OI} = \mu d/(1-\mu^2)$, and $\overline{EI} = \mu\overline{PI}$. Substitution into (22) the following is obtained:

$$\begin{aligned}\left(\tfrac{\mu d}{1-\mu^2}\right)^2 &= \left(\tfrac{\mu^2 d}{1-\mu^2}\right)^2 + (\mu\overline{PI})^2 \\ &\quad - 2\left(\tfrac{\mu^2 d}{1-\mu^2}\right)(\mu\overline{PI})\cos(\psi_E - \theta_E)\end{aligned} \tag{23}$$

Bringing all the terms to the right hand side and simplifying (23),

$$0 = \overline{PI}^2 - \tfrac{2\mu d}{1-\mu^2}\cos(\psi_E - \theta_E)\overline{PI} - \tfrac{d^2}{1-\mu^2} \tag{24}$$

Using the quadratic equation, $\overline{PI}$ may be solved in terms of the evader-pursuer speed ratio and the pursuer-evader distance using (23). Letting

$$a = 1, \quad b = -\tfrac{2\mu d\cos(\psi_E - \theta_E)}{1-\mu^2}, \quad c = -\tfrac{d^2}{1-\mu^2}$$

The distance $\overline{PI} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Substitution of a, b, and c, into the quadratic formula the distance $\overline{PI}$ can be solved. The positive case is where the pursuer moves forward and captures the evader, while the negative case is where the pursuer moves backward and intersects the Apollonius circle at a point of no interest. The quadratic equation for the solution of $\overline{PI}$ is the following:

$$\overline{PI} = \tfrac{\mu d\cos(\psi_E - \theta_E)}{(1-\mu^2)} + d\sqrt{\left(\tfrac{\mu\cos(\psi_E - \theta_E)}{(1-\mu^2)}\right)^2 + \tfrac{1}{1-\mu^2}} \tag{25}$$

Since $v_P = 1$, $t_{\text{go}} = d(\sigma_1 + \sqrt{\sigma_1^2 + \sigma_2})$ where $\sigma_1 = \mu\cos(\psi_E - \theta_E)/(1-\mu^2)$ and $\sigma_2 = 1/(1-\mu^2)$. ■

C. Optimal Control Problem, B - Defender Strategy

Lemma 4. *The optimal strategy for the defender is a straight-line constant-heading strategy.*

Proof. Evaluating (12) using the Hamiltonian, $\mathscr{H}_B$ from (10) the costates dynamics are as follows:

$$\dot{p}^*_{x_P,B} = \dot{p}^*_{y_P,B} = \dot{p}^*_{x_D,B} = \dot{p}^*_{y_D,B} = 0 \tag{26}$$

The costates for each individual optimization problem have no dynamics and are constant under optimal play.

Taking the partial in (13) using the Hamiltonian in (10) with respect to the control of the defender ψ_D, the following is obtained:

$$0 = -p^*_{x_D,B}\alpha\sin\psi^*_D + p^*_{y_D,B}\alpha\cos\psi^*_D \tag{27}$$

and therefore as derived in Lemma 1,

$$\psi^*_D = \sin^{-1}\left(\frac{p^*_{y_D,B}}{\sqrt{p^{*2}_{x_D,B} + p^{*2}_{y_D,B}}}\right) \tag{28}$$

Since the costates are constant, the heading of the pursuer is constant under optimal play. ■

Evaluating (13) using the Hamiltonian, $\mathscr{H}_A$ from (9) the optimal control is found to depend solely upon the costates and the parameter μ. Similarly, evaluating (13) using the Hamiltonian, $\mathscr{H}_B$ from (10) the optimal control is found to depend solely upon the costates and the parameter α. Since the optimal costates are constant, it may be inferred that the optimal control for the defender and the pursuer are also constant; hence, all optimal strategies are straight-line trajectories. Because all optimal strategies for the pursuer and defender are straight line trajectories, the optimal heading for the pursuer is described using the geometry of Apollonius, and the optimal strategy for the Defender is that of the maximum-time observation from [17].

Three scenarios of interest are examined: When $t_{\text{go}} \geq t_{\text{exp}}$, when $t_{\text{go}} < t_{\text{exp}}$, and when $t_{\text{exp}} = 0$.

1) Time-to-go greater than or equal to exposure time: A figure which describes the engagement when the time-to-go (t_{go}) is greater than or equal to the maximum possible exposure time (t_{exp}) is shown in Figure 1.

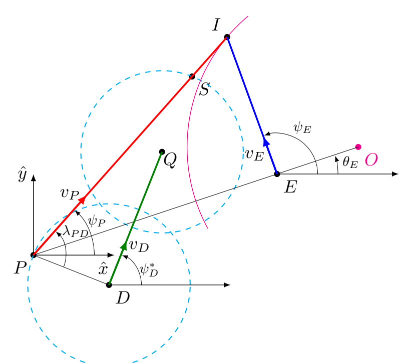

Fig. 1. The directed energy defense scenario wherein the maximum possible exposure time is less than the time-to-go.

From the figure, the interception point made by the pursuer and evader occurs after the pursuer escapes the EZ of the defender; this means that the heading taken by the defender is one which maximizes the time that the pursuer is inside his EZ.

Lemma 5. *Suppose that scenario terminates in P exiting the EZ prior to capturing E, then the optimal heading of the defender is*

$$\psi^*_D = \cos^{-1}\left(\tfrac{(\alpha^2-1)\sin\lambda_{PD}}{\alpha^2 + 2\alpha\cos\lambda_{PD} + 1}\right), \tag{29}$$

where α is the speed ratio between the defender and the pursuer and λ_{PD} is the line of sight angle (positive counter clockwise) from the pursuer to the defender.

Proof. See Theorem 1 from [17]. ■

2) Exposure time greater than time-to-go: Consider the case when the time-to-go is less than the maximum possible exposure time; this case is illustrated in Figure 2. Since the defender's optimal strategy provides the maximum possible exposure of the pursuer, the question which needs to be answered is, "What heading bounds provide an exposure time of at least time-to-go?" Specifically, what headings ψ_{D_1} and ψ_{D_2} provide exactly t_{go} exposure time?

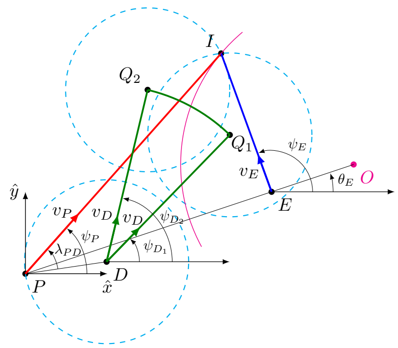

Fig. 2. The directed energy defense scenario wherein the maximum possible exposure time is greater than the time-to-go. In this case, suboptimal headings may be used to provide an exposure time equal to the time-to-go.

In order to solve for the defender headings which provide exactly t_{go} exposure time, when the maximum possible exposure provided by the optimal heading from (29) is implemented, one begins with locating the interception point with respect to the defender's initial location. Where necessary, the subscript $[D]$ is used to represent a point measured from the defender's initial position, D. For example, the point I with respect to D is $I_{[D]} = (x_{I[D]}, y_{I[D]})$; more explicitly,

$$x_{I[D]} = x_I - x_D, \quad y_{I[D]} \quad = y_I - y_D \tag{30}$$

The location of the defender at time-to-go with respect to D is $Q_{[D]} = (x_{Q[D]}, y_{Q[D]})$. Furthermore, the range between the defender and the pursuer at the time-to-go is R_D, and therefore

$$R_D^2 = \left(x_{I[D]} - x_{Q[D]}\right)^2 + \left(y_{I[D]} - y_{Q[D]}\right)^2 \tag{31}$$

Recognizing that the speed ratio between the defender and the pursuer is α, the distance traversed by the pursuer is $\overline{PI}$ and the distance traversed by the defender is $\alpha\overline{PI}$. Therefore the location of the defender at the time-to-go is

$$Q_{[D]} = \alpha\overline{PI}\cos\psi_D\hat{x} + \alpha\overline{PI}\sin\psi_D\hat{y} \tag{32}$$

In the pursuer global frame located at P, the location of the defender at the time-to-go is

$$Q = \alpha\overline{PI}\cos\psi_D\hat{x} + \alpha\overline{PI}\sin\psi_D\hat{y} + x_D\hat{x} + y_D\hat{y} \tag{33}$$

Substitution of (32) into (31) the following is obtained

$$\begin{aligned} R_D^2 = &\left(x_{I[D]} - \alpha\overline{PI}\cos\psi_D\right)^2 \\ &+ \left(y_{I[D]} - \alpha\overline{PI}\sin\psi_D\right)^2 \end{aligned} \tag{34}$$

Expanding (34) and collecting terms in ψ_D,

$$\begin{aligned} R_D^2 =& x_{I[D]}^2 + y_{I[D]}^2 + \alpha^2\overline{PI}^2 \\ &- 2\alpha\overline{PI}x_{I[D]}\cos\psi_D - 2\alpha\overline{PI}y_{I[D]}\sin\psi_D \end{aligned} \tag{35}$$

Notice that (35) has the form $A\cos\psi_D + B\sin\psi_D + C = 0$, where

$$\begin{aligned} A &= -2\alpha\overline{PI}x_{I[D]} \\ B &= -2\alpha\overline{PI}y_{I[D]} \\ C &= x_{I[D]}^2 + y_{I[D]}^2 + \alpha^2\overline{PI}^2 - R_D^2 \end{aligned} \tag{36}$$

Using the trigonometric half-angle formula, $\sin\psi_D$ and $\cos\psi_D$ may be re-written as a function of $\tan\psi_D$. The identity is repeated here for the reader's convenience.

$$\cos\psi_D = \tfrac{1-\tan^2(\psi_D/2)}{1+\tan^2(\psi_D/2)}, \quad \sin\psi_D = \tfrac{2\tan(\psi_D/2)}{1+\tan^2(\psi_D/2)}$$

Let $\tau = \tan(\psi_D/2)$, then (36) may be re-written in terms of τ.

$$\begin{aligned} 0 &= A\tfrac{1-\tau^2}{1+\tau^2} + B\tfrac{2\tau}{1+\tau^2} + C \\ &= A(1-\tau^2) + 2B\tau + C(1+\tau^2) \\ &= (C-A)\tau^2 + 2B\tau + (A+C) \end{aligned} \tag{37}$$

Using the quadratic formula:

$$\tau = \tfrac{-2B\pm\sqrt{4B^2-4(C-A)(A+C)}}{2(C-A)} = \tfrac{-B\pm\sqrt{B^2-C^2+A^2}}{C-A} \tag{38}$$

Therefore, because of the quadratic equation, two solutions (as expected) for ψ_D exist.

$$\begin{aligned} \psi_{D_1} &= 2\arctan\left(\tfrac{-B+\sqrt{B^2-C^2+A^2}}{C-A}\right) \\ \psi_{D_2} &= 2\arctan\left(\tfrac{-B-\sqrt{B^2-C^2+A^2}}{C-A}\right) \end{aligned} \tag{39}$$

Axiom 1. *When the exposure time is greater than the time-to-go all headings in the closed interval $\psi_D \in [\psi_{D_1}, \psi_{D_2}]$ provide a final time of time-to-go.*

Axiom 2. *The maximum exposure time (t_{exp}), as described in [17], as a function of the line of sight angle, λ_{PD}, the speed ratio between the defender and the pursuer, α, and radius of the EZ, R_D is*

$$t_{exp} = 2R(\alpha + \cos\lambda_{PD})/(1-\alpha^2). \tag{40}$$

Axiom 3. *The line of sight angle, λ_{PD}, takes on angles from $-180\,\text{deg}$ to $180\,\text{deg}$. The exposure time (t_{exp}) is as described in (40). For all possible values of λ_{PD}, the maximum possible exposure time occurs when $\lambda_{PD} = 0$;*

$$\bar{t}_{exp} = \max_{\lambda_{PD}} \tfrac{2R(\alpha+\cos\lambda_{PD})}{1-\alpha^2} = \tfrac{2R(\alpha+1)}{1-\alpha^2} \tag{41}$$

Axiom 4. *Setting (40) equal to zero, the line of sight angles by which exposure is not possible are in the interval*

$$\lambda_{PD} \in [-\pi, -\cos^{-1}(-\alpha)] \cup [\cos^{-1}(-\alpha), \pi] \tag{42}$$

Axiom 5. *The final time is limited when the evader is captured by the pursuer before the pursuer escapes the defender's EZ; this occurs when $\bar{t}_{exp} > t_{go}$. In the event that $2R(\alpha+1)/(1-\alpha^2) > t_{go}$ the line-of-sight angle where $t_{go} = t_{exp}$ occurs at the angle $\lambda_{PD,go}$.*

$$\lambda_{PD,go} \triangleq \cos^{-1}\left(\tfrac{(1-\alpha^2)t_{go}}{2R} - \alpha\right) \tag{43}$$

III. Main Results

From the defender's perspective, the optimal control problem ends when either the pursuer captures the evader or when the pursuer escapes the EZ of the defender, $\mathscr{C} = \mathscr{C}_A \cup \mathscr{C}_B$. In cases where the line of sight angle λ_{PD} is as described in (42), the defender terminates immediately, as it is unable to keep the pursuer in his EZ for any amount of time. From the pursuer's perspective, the optimal control problem ends when he captures the evader, $\mathscr{C}_B$.

A. Exposure Time

Theorem 1. *Given the optimal control problem specified by (2), (4) and (7) the optimal final time t_f^* is*

$$t_f^* = \begin{cases} t_{go} & \lambda_{PD} \in \Lambda_{go} \\ t_{exp} \textit{ from } (40) & \lambda_{PD} \in \Lambda_* \\ 0 & \lambda_{PD} \in \Lambda_0 \end{cases} \tag{44}$$

where

$$\Lambda_0 = \left\{ \lambda_{PD} \middle| \begin{array}{c} -\pi \leq \lambda_{PD} \leq -\arccos(-\alpha), \\ \arccos(-\alpha) \leq \lambda_{PD} \leq \pi \end{array} \right\}$$

$$\Lambda_{go} = \left\{ \lambda_{PD} \middle| \lambda_{PD} \leq \left| \arccos\left(\frac{(1-\alpha^2)t_{go}}{2R} - \alpha \right) \right| \right\}$$

$$\Lambda_* = \left\{ \lambda_{PD} \middle| \lambda_{PD} \notin \Lambda_0 \textit{ and } \lambda_{PD} \notin \Lambda_{go} \right\}$$

Proof. Three scenarios are possible:

1) P captures E before escaping D's EZ.
In the first case, $\lambda_{PD} \in \Lambda_{go}$ implies that P captures E before escaping D's EZ. Therefore, from Lemma 3 $t_{\text{go}} < t_{\text{exp}}$. Therefore, $t_f^* = t_{\text{go}}$ by Axiom 5.
2) P captures E after escaping D's EZ.
In the second case, $\lambda_{PD} \in \Lambda_*$ implies that P captures E after escaping or lies on the border of D's EZ. Therefore, from Axiom 2, $t_f^* = t_{exp}$ from (40).
3) D is incapable of exposing P for any amount of time.
$\lambda_{PD} \in \Lambda_0$ implies that D is unable to expose P for any amount of time and therefore $t_f^* = 0$ by Axiom 4.

■

Corollary 1. *In the event that $\bar{t}_{exp} < t_{go}$, Λ_{go} is empty.*

Proof. $\bar{t}_{exp}$ is the maximum possible time that the pursuer is contained in the EZ of the defender by Axiom 3. If $\lambda_{PD} \in \Lambda_{go}$ then by axiom 2:

$$t_{\text{exp}} = 2R(\alpha + \cos\lambda_{PD})/(1-\alpha^2)$$

and by the definition of Λ_{go}

$$t_{\text{exp}} \geq \frac{2R}{1-\alpha^2}\left(\alpha + \left(\frac{(1-\alpha^2)t_{\text{go}}}{2R} - \alpha\right)\right) \tag{45}$$

expanding:

$$t_{\text{exp}} \geq \frac{2R\alpha}{1-\alpha^2} + \frac{2R}{1-\alpha^2}\frac{(1-\alpha^2)t_{\text{go}}}{2R} - \frac{2R\alpha}{1-\alpha^2} \tag{46}$$

And therefore $t_{\text{exp}} \geq t_{go}$. By contradiction, in order for λ_{PD} to be an element of Λ_{go}, t_{exp} must be greater than t_{go}; but, by our assertion, $\bar{t}_{exp} < t_{\text{go}}$. ■

B. Defender Strategy

Theorem 2. *The Defender's strategy depends upon the initial locations of the agents as well as the problem parameters: α, μ, λ_{PD}, and ψ_E. In the event that the defender in unable to expose the pursuer for any amount of time, e.g. $t_{exp} = 0$ no matter the heading that ψ_D should take, the defender's strategy is of no consequence. However, for exposure times greater than zero, the defender's choice of heading is*

$$\psi_D^* = \begin{cases} \{\psi_D | \psi_D \in [\psi_{D_1}, \psi_{D_2}]\} & \lambda_{PD} \in \Lambda_{go} \\ \arccos\left(\frac{(\alpha^2-1)\sin\lambda_{PD}}{\alpha^2+2\alpha\cos\lambda_{PD}+1}\right) & \lambda_{PD} \in \Lambda_* \\ \textit{undefined} & \lambda_{PD} \in \Lambda_0 \end{cases} \tag{47}$$

Proof. Three scenarios are possible

1) P captures E before escaping D's EZ
In the first case, $\lambda_{PD} \in \Lambda_{go}$ implies that P captures E before escaping D's EZ. Therefore by Axiom 1, $\psi_D^* = \{\psi_D | \psi_D \in [\psi_{D_1}, \psi_{D_2}]\}$.
2) P captures E after escaping D's EZ
In the second case, $\lambda_{PD} \in \Lambda_*$ implies that P captures E after escaping D's EZ. Therefore from Lemma 5, $\psi_D^* = \arccos\left(\frac{(\alpha^2-1)\sin\lambda_{PD}}{\alpha^2+2\alpha\cos\lambda_{PD}+1}\right)$
3) D is incapable of exposing P for any amount of time.
$\psi_{DP} \in \Lambda_0$ implies that D is unable to expose P for any amount of time by Axiom 4.

■

IV. Examples

Consider the directed energy defense scenario with the EZ radius of 2km, the speed ratio between the evader and the pursuer is $\mu = 0.5$ and the speed ratio between the defender and the pursuer is $\alpha = 0.6$. The evader takes a heading of 110 degrees from East. In order to highlight the defender strategy in (47), two cases are considered: $\lambda_{PD} = -70$deg and $\lambda_{PD} = -40$deg. Common to the examples, the pursuer is located at the origin, $P = (0, 0)$, and the evader is located at $E = (6, 2)$.

The first step in both examples is to construct the Apollonius circle and determine the heading of the pursuer as well as the time-to-go. Using (20), the optimal pursuer strategy which intercepts the evader in minimum time is found to be 48.4 deg. from East. Using (21) the time-to-go is found to be 7.189 sec. The maximum possible exposure time, independent of the bearing of the defender from the pursuer is found using (41); $\bar{t}_{exp} = 10.000$ sec. Also, the sets $\Lambda_0, \Lambda_{exp}, \Lambda_{go}$, and Λ_* are found to be:

$$\begin{aligned} \Lambda_0 &= [-180, -126.870]\text{deg} \cup [126.870, 180]\text{deg} \\ \Lambda_{go} &= [-56.620, 56.6120]\text{deg} \\ \Lambda_* &= (-126.870, -56.620]\text{deg} \cup [56.620, 126.870)\text{deg} \end{aligned} .$$

A. Example 1: λ_{PD} = -70 deg

The first example highlights the case where the exposure time is less than the time-to-go. For this example, consider the defender to be 70 deg starboard from the pursuer's heading and 2km from the pursuer – at the onset the pursuer is inside the defender's EZ. A figure which describes this example is shown in Figure 3.

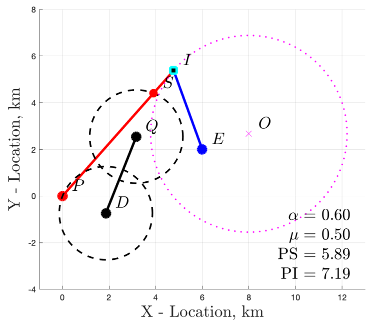

Fig. 3. The directed energy defense scenario where the bearing from the pursuer to the defender is 70 deg starboard. The speed ratio between the evader to the pursuer is $\mu = 0.50$ and the speed ratio between the defender and the pursuer is $\alpha = 0.60$.

In Figure 3, the solid red line represents the pursuer's course, the solid black line represents the defender's course, and the solid blue line represents the evader's course. The points P, E, and D represent the initial location of the pursuer, evader, and defender respectively. The points S represent the location of the pursuer when he escapes the defender's EZ, who is located at the point Q. The point I represents the capture location of the evader by the pursuer which is dictated by the Apollonius circle centered at the point O.

In this example, the angle $\lambda_{PD} = -70$ deg $\in \Lambda_*$ and the exposure time and optimal heading are found using (44) and (47) respectively. $t_{\text{exp}} = 5.888$ sec and $\psi_D = 68.281$ deg from East.

From this scenario, the defender heads toward the pursuer in order to maximize the time that the pursuer stays inside his EZ. If the defender were to deviate from this optimal heading (ψ_D^*), the time that the pursuer is exposed would be less than the calculated 5.888 sec. Since the pursuer does not capture the evader before the defender loses contact with the pursuer, the defender has a unique optimal heading that maximizes the time that the pursuer is inside his EZ.

B. Example 2: λ_{PD} = -40 deg

The second example highlights the case where the exposure time is greater than the time-to-go. For this example, consider the defender to be 40 deg starboard from the pursuer's heading and 2km from the pursuer. A figure which describes this example is shown in Figure 4. In Figure 4, the solid red line represents the pursuer's course, the solid black line represents the defender's course, and the solid blue line represents the evader's course. The points P, E, and D represent the initial location of the pursuer, evader, and defender respectively. The point I represents the capture location of the evader by the pursuer which is dictated by the Apollonius circle centered at the point O. Since the pursuer is inside the defender's EZ for the entire engagement, there exists an interval of headings which the defender can take which ensure that he is contained in his EZ for the entirety

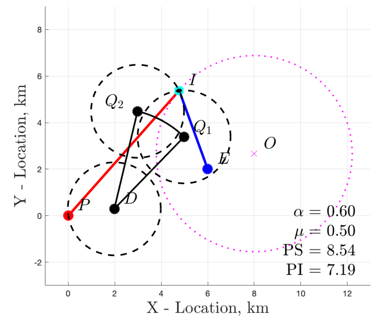

Fig. 4. The directed energy defense scenario where the bearing from the pursuer to the defender is 40 deg starboard. The speed ratio between the evader to the pursuer is $\mu = 0.50$ and the speed ratio between the defender and the pursuer is $\alpha = 0.60$.

of the engagement until the pursuer captures the evader. The limiting headings are shown by the arc between Q_2 and Q_1.

In this example, the angle $\lambda_{PD} = -40$ deg $\in \Lambda_{go}$ and the exposure time and optimal heading are found using (44) and (47) respectively. $t_{\text{exp}} = t_{\text{go}} = 7.189$ sec and using (39) the limiting headings $\psi_{D_1} = 45.869$ deg and $\psi_{D_2} = 76.584$ deg from East. This means, that if the defender takes any heading between 45.869 and 76.584 deg, the pursuer will stay inside the EZ for the entirety of the engagement.

Next, consider every possible line of sight angle from the pursuer to the defender, $\lambda_{PD} \in [-180, 180]$ deg. Assuming that the defender implements the strategy described in (47) and the pursuer captures the evader in minimum time using the optimal heading in (20), the exposure time as a function of the line of sight angle λ_{PD} is shown in Figure 5.

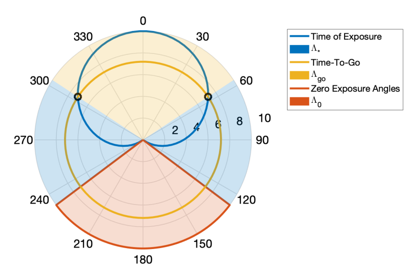

Fig. 5. The exposure time as a function of the line of sight angle λ_{PD} is depicted in this polar plot when $E = (6, 2)$, $\alpha = 0.6$, and $\mu = 0.50$. This polar plot is a graphical representation of (44)

In Figure 5 the blue Limaçon describes the time that the pursuer could remain inside the EZ of the defender if the evader were not captured by the pursuer. The yellow circle represents the time-to-go – the evader is captured by the pursuer. The red lines describe the line of sight headings by which the pursuer is unable to be contained inside the EZ of the defender regardless of the defender's strategy. The black circles represent the angle and time at which the pursuer is inside the defender's EZ for exactly the same time as the time-to-go.

In Figure 6 the strategy of the defender as a function of the line of sight angle λ_{PD} is shown.

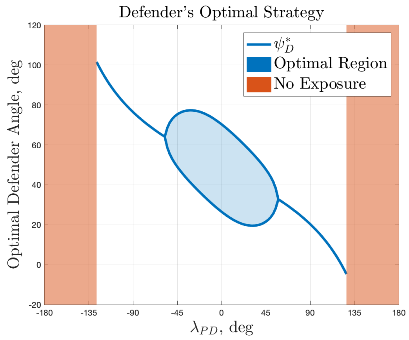

Fig. 6. The defender's heading (ψ_D) as a function of the line of sight angle λ_{PD} as described in (47). The case presented is that of the examples where $E = (6, 2)$, $\alpha = 0.6$, and $\mu = 0.50$.

In Figure 6 the red regions represent line of sight angles where the defender is unable to contain the pursuer for any amount of time. The blue line represents the optimal strategy of the defender. The shaded blue region represents the cases where the time-to-go limits the amount of time that the pursuer is contained in the defender's EZ.

An interesting observation about the shaded blue region in Figure 6 is that the shaded region is widest at $\lambda_{PD} = 0$. This is because the maximum possible exposure time $\bar{t}_{exp}$ occurs at $\lambda_{PD} = 0$ and because the pursuer captures the evader before it can escape the EZ of the defender. Also, due to the fact that the amount of time that the defender can keep the pursuer inside his EZ decreases as he deviates from the optimal heading ψ_D^*, the amount of deviation when $\lambda_{PD} = 0$ is a maximum; thus, the blue shaded region is the widest when $\lambda_{PD} = 0$. The shaded region collapses to a unique heading at the angle $\pm\lambda_{PD,go} = \pm 56.620$ deg as described in (43). This means that for headings $\lambda_{PD} \in \Lambda_*$ the defender's strategy is unique and is ψ_D^* as in (29).

V. Conclusions

In conclusion, the directed energy defense of a non-maneuvering evader against an incoming threat has been presented. Making use the optimal control theory, two optimization problems are posed and solved in tandem. First, the min-time capture of a non-maneuverable evader and then the max-time exposure of the pursuer by a defender with circular engagement zone. From the costates, the optimal trajectories of the pursuer and the defender were shown to be straight line trajectories. Leveraging the optimal observer strategy from [17] for the defender, the optimal defender strategy and exposure time is found in closed form.

Two examples are presented in order to demonstrate the intricacies surrounding the target defense scenario. The first example demonstrates the defender's optimal strategy when the pursuer captures the evader after escaping the defender's EZ and second demonstrates the defender's optimal strategy when the pursuer captures the evader before it can escape the defender's EZ. Also presented are conditions for which the line-of-sight angle limits the time of exposure – the exposure time is zero, independent of the defender's chosen strategy or limited by the time-to-go.

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